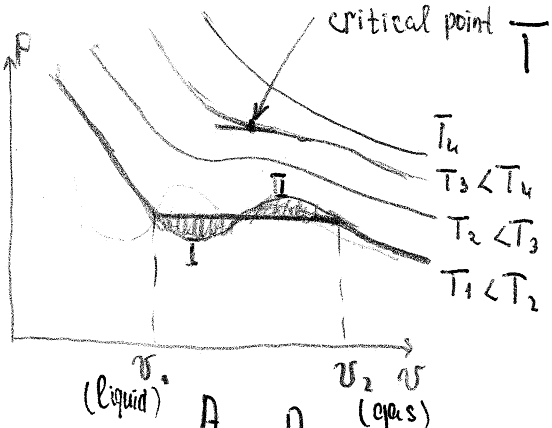
Lecture 1

Keywords: Scaling concept; Critical phenomena; Soft Condensed Matter.

Collecting behaviour of large systems.

$\boxed{\text{Van der Waals theory.}}$

$$\left(p + \frac{a}{v^2}\right)(v - b) = T \qquad v = \frac{V}{N} \qquad pv = T$$



$A_I = A_{II}$

$T = \text{const}$

$$-\left(\frac{\partial p}{\partial v}\right)_T < 0$$

$$\left[\text{inequality } \left(\frac{\partial p}{\partial v}\right)_T \leq 0\right]$$

for critical point:

$$\begin{cases} \left(\dfrac{\partial p}{\partial v}\right)_{T_c} = 0 \\ \left(\dfrac{\partial^2 p}{\partial v^2}\right)_{T_c} = 0 \end{cases}$$

$$T_c = \frac{8a}{27b} \qquad p_c = \frac{a}{27b^2} \qquad v_c = 3b$$

$$\tilde{T} = \frac{T}{T_c} \qquad \tilde{p} = \frac{p}{p_c} \qquad \tilde{v} = \frac{v}{v_c}$$

$$\boxed{\left(\tilde{p} + \frac{3}{\tilde{v}^2}\right)(3\tilde{v} - 1) = 8\tilde{T}}$$

$$\tau = \frac{T - T_c}{T_c} \qquad \tau \to 0 \qquad \tau < 0 \qquad \boxed{\rho_1 - \rho_2 = A\,\tau^\beta} \qquad \beta > 0$$

$$\varkappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \qquad \varkappa = \begin{cases} B'(-\tau)^{-\gamma'}, & \tau < 0 \\ B\,\tau^{-\gamma}, & \tau > 0 \end{cases}$$

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_T \qquad C_V = \begin{cases} C'(-\tau)^{-\alpha_1}, & \tau < 0 \\ C\cdot\tau^{-\alpha}, & \tau > 0 \end{cases}$$

$$\tau = 0 \qquad p - p_c = F\,|\rho - \rho_c|^\delta$$

$$\boxed{\alpha' + 2\beta + \gamma' \geq 2} \quad \text{Rushbrook}$$

$$\boxed{\alpha' + \beta(1 + \delta) \geq 2} \quad \text{Griffith}$$

$\tilde{p} = 1 + \pi \quad \tilde{v} = 1 + \omega \quad \tilde{\tau} = 1 + \tau$

$|\pi| \ll 1 \quad |\omega| \ll 1 \quad |\tau| \ll 1$

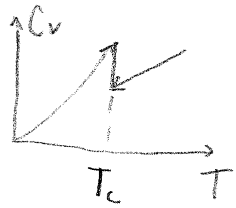
$\pi = +4\tau - 6\tau\omega - \frac{3}{2}\omega^3$

$\tau = 0 \rightarrow \delta = 3$ ②

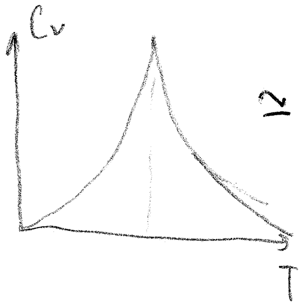
$\pi = 0 \quad \tau < 0 \rightarrow \omega^2 \sim -\tau \quad \beta = 1/2$

$\gamma = \gamma' = 1$

$\alpha = \alpha' = 0$



All experimental investigations show that these numbers are wrong.



$\simeq -\ln|T - T_c|$ not finite jump

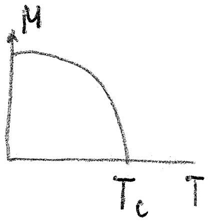
$-\ln x \quad \frac{1}{x^{0.1}}$ not distinguished in experiment

$\beta \simeq 1/3 \qquad \gamma = \gamma' \simeq 4/3 \qquad \delta \simeq 5$

Interactions in different liquids are different. Hower, the behavior of all exponents is the same and it differs from Van der Waals theory.

Feromagnet - paramagnet

Fe_3O_4



Curie

$M = \chi H$

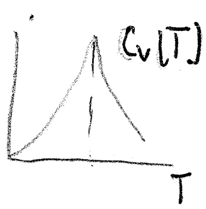
$M \leftrightarrow \rho_1 - \rho_2$

$H \leftrightarrow p - p_c$

$\chi \leftrightarrow \varkappa$

$\tau = \frac{T - T_c}{T}$

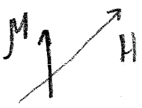
$\tau \rightarrow 0 \quad M(\tau) \sim (-\tau)^{\beta}, \quad \tau < 0$



$-\ln|T - T_c|$

$\beta \simeq 1/3 \qquad \gamma = \gamma' \simeq 4/3 \qquad \tau = 0 \quad \boxed{M \sim H^{1/\delta}} \quad \delta \simeq 5$

Mean-field theory in FM



$U = -\vec{\mu}\vec{H}$

$S \quad \mu = \frac{M}{S}S, \ \frac{M}{S}(S-1), \ldots 0, \ldots, -\frac{M}{S}S$

$2S+1$

$S = 1/2 \quad \mu = -\mu_B, \mu_B$

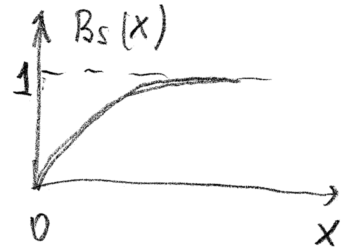
$S = 1 \quad \mu = -\mu_B, 0, \mu_B$

$$\langle M_z \rangle = \sum_{M=-S}^{S} \exp\left(-\frac{U_M}{T}\right) M \Big/ \sum_{M=-S}^{S} \exp\left(-\frac{U_M}{T}\right) \qquad U_M = \frac{\mu H}{S} M$$

$$\sum_{M=-S}^{S} e^{xM} M = \frac{d}{dx} \sum_{M=-S}^{S} e^{xM}$$

$$\underline{\langle M \rangle = \langle S^z \rangle = S\, B_S\left(\frac{\mu H}{T}\right)}$$ Brilluen function

$$B_S(x) = \frac{2S+1}{2S} \,\mathrm{cotant}\left(\frac{2S+1}{2S} x\right) - \frac{1}{2S} \,\mathrm{cotanh}\left(\frac{x}{2S}\right)$$



Weiss have discovered ferromagnetism

Each spin' (Weiss didn't know about spins) is subjected to other magnetic particles ⇒ in their field λ

$$H \to H + \lambda \langle S_z \rangle \qquad T < T_c = \frac{\lambda S(S+1)}{3}$$

In this theory $\beta \simeq 1/2$ $\quad \gamma = \gamma' = 1$ $\quad \delta = 3$ $\quad \alpha = \alpha' = 0$

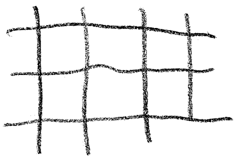
the same as in Van der Waals.

Ferromagnet - paramagnet is quantum phenomenon; liquid-gas transition is classical phenomenon. But they have the same exponents ...

Lecture 2

Ising model (3d example)

Model of binary alloy.



A – B
Cu – Au
FCC

low temperature

Let's assume that only nearest neighbours interact.

A — A V_{AA} A — B V_{AB} B — B V_{BB}

$\mathcal{H}(Q)$ $Z = \sum_{\{Q\}} e^{-\beta \mathcal{H}(Q)}$ partition function

$\beta = \frac{1}{T}$

$F = -\frac{1}{\beta} \ln Z$

$\boxed{\mathcal{H} = \sum_{\langle ij \rangle} V_{ij}}$ $\quad \epsilon_i = \begin{cases} +1, & i = A \\ -1, & i = B \end{cases}$ $\langle ij \rangle$ – nearest pairs

$\boxed{V_{ij} = a\epsilon_i\epsilon_j + b(\epsilon_i + \epsilon_j) + c}$

$V_{AA} = a + 2b + c \qquad V_{BB} = a - 2b + c \qquad V_{AB} = -a + c$

$2a = \frac{V_{AA} + V_{BB}}{2} - V_{AB} \qquad b = \frac{V_{AA} - V_{BB}}{2}$

$\mathcal{H} = -J \sum_{\langle ij \rangle} \epsilon_i \epsilon_j - h \sum_i \epsilon_i$

$J > 0$ atoms prefare to form A — A B — B

$J < 0$ atoms prefare A — B — A

We suppose that it'll be ordering in low temperature.

Why? Minimization of free energy $\boxed{F = E - TS}$

At high enough temperatures S high TS - main contribution
In low T system will try to decrease F by ordering.
The main contribution of Ising to physics is this model, he solved it for 1-dim. case when he was a <u>student</u> of Hamburg university.

Let's introduce <u>mean-field theory</u> to this problem:
to decrease the number of variables (degrees of freedom) 2^N (+1 or -1)
Replace dynamical variable (+1 or -1) by its <u>average</u>.

$$\sigma_i \sigma_j \to \sigma_i \langle \sigma_j \rangle + \langle \sigma_i \rangle \sigma_j - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

$\langle \sigma_i \sigma_j \rangle = \langle \sigma_i \rangle \langle \sigma_j \rangle$ independent variables

$\langle \sigma_i \rangle = \langle \sigma \rangle \qquad \mathcal{H}_i = -h_{eff}\, \sigma_i \qquad h_{eff} = 2J\langle \sigma \rangle + h$

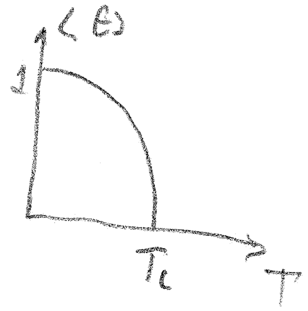
$$\langle \sigma \rangle = \frac{\sum_{\sigma=\pm 1} \sigma \exp(\beta h_{eff} \sigma)}{\sum_{\sigma=\pm 1} \exp(\beta h_{eff} \sigma)} = \tanh(\beta h_{eff}) = \tanh(\beta(h + 2J\langle \sigma \rangle))$$

$S = 1/2 \rightarrow$ Brillouin function.

$h = 0 \qquad 2J\beta_c = 1 \qquad T_c = 2J \qquad \langle \sigma \rangle = 0$

$T < T_c \qquad \langle \sigma \rangle \neq 0$

$T > T_c \quad h \to 0 \qquad \langle \sigma \rangle = \chi h$



$\langle \sigma \rangle \sim (T_c - T)^{\beta} \qquad \underline{\beta = 1/2} \qquad \underline{\gamma = \gamma' = 1}$

$\chi \sim \frac{1}{T - T_c},\ T > T_c \qquad \underline{\alpha = \alpha' = 0} \qquad \underline{\delta = 3} \qquad T = T_c \ \langle \sigma \rangle^3 \sim h$

Exact solution of Ising model in 1D case

interact with energy J

$$\mathcal{H} = -J\sum_i \epsilon_i \epsilon_j - h\sum_i \epsilon_i \qquad Z = \sum_{\substack{\epsilon_j=\pm1 \\ \epsilon_i=\pm1}} \exp(-\beta\mathcal{H})$$

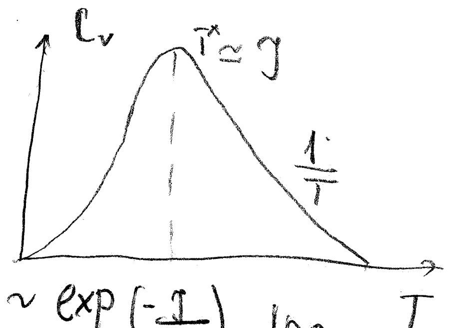
for $\underline{h=0}$ very simple $\sum_{\epsilon_N=\pm1} \exp(\beta J \epsilon_N \epsilon_{N-1}) = 2\cosh\beta J$

$$\exp(\beta J \epsilon_{N-1}\epsilon_{N-2}) = 2\cosh\beta J$$

$$Z = (2\cosh\beta J)^N$$

$$F = -\frac{1}{\beta}\ln Z = -\frac{N}{\beta}\ln(2\cosh\beta J)$$

$$S = -\frac{\partial F}{\partial T} \qquad C_v = T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$$



$\sim \exp(-\frac{J}{T})$ **no jump!**

for $h \neq 0$

$\epsilon_{N+1} = +1, -1$

$Z_N(\epsilon_{N+1})$ – partition function for N atoms assuming that $\epsilon_{N+1} = +1$, or -1

$$Z_{N+1}(\epsilon_{N+2}) = \sum_{\epsilon_{N+1}=\pm1} Z_N(\epsilon_{N+1})\exp[\beta J \epsilon_{N+1}\epsilon_{N+2} + \beta h\,\epsilon_{N+1}] \quad \text{exact}$$

Let's solve this equation (not exact) for thermodynamic limit $N\to\infty$

$$\boxed{Z_N = e^{-\beta F} \simeq e^{\beta \mu N}}\ \text{exact}, \qquad \boxed{Z_{N+1} = e^{\beta\mu} Z_N},\ N\to\infty \quad \leftarrow \text{not exact}$$

$F \sim N \qquad F \equiv -\mu N$, μ – chemical potential

$$e^{\beta\mu} Z_N(x) = \sum_y Z_N(x)\exp[\beta J x y + \beta h y]$$

$$\hat{M} = \begin{vmatrix} e^{\beta J + \beta h} & e^{-\beta J - \beta h} \\ e^{-\beta J + \beta h} & e^{\beta J - \beta h} \end{vmatrix} \qquad e^{\beta\mu} = e^{\beta J}\cosh\beta h + e^{-\beta J}\sqrt{1 + e^{4\beta J}\cosh^2\beta h}$$

$$M = \langle\epsilon\rangle = -\frac{\partial F}{\partial h} \qquad \chi = \frac{\partial\langle M\rangle}{\partial h} = -\frac{\partial^2 F}{\partial h^2}$$

The result is drastically different from mean-field, there is no critical point at all! $h \to 0$ $\langle \sigma \rangle = 0$, $T \neq 0$,

$M = \langle \sigma \rangle = 1$, $T = 0$,

$+ + + + | - - - - \ldots$

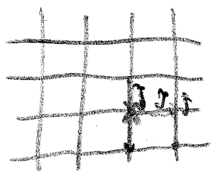
$E \simeq 2J$ Statistical weight of N boundaries $S \simeq N \ln N$

$E_N \simeq 2JN$ $F \simeq JN - TN \ln N$ at any finite temperature

it is favourable to create these boundaries. $N \simeq \exp\left(-\frac{J}{T}\right)$

Only at $T = 0$ system ~~has~~ no defects, is ordered

Two-dimensional Ising model.



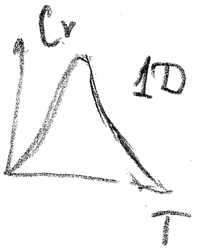
$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \qquad Z = \sum_{\sigma_i = \pm 1} e^{-\beta \mathcal{H}}$$

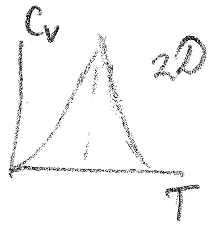
$N \to \infty$

L. Onsager 1944 $\qquad F = -\frac{1}{\beta} \ln Z$

$$\frac{\beta F}{N} = -\ln Z = \frac{1}{2} \int_0^{2\pi} \frac{d\xi}{2\pi} \int_0^{2\pi} \frac{d\eta}{2\pi} \ln\left[(\cosh 2\beta J)^2 - (\sinh 2\beta J)^2 (\cos\xi + \cos\eta)\right]$$

$\tanh \frac{J}{T_c} = \sqrt{2} - 1$ $\qquad \tau = \frac{T}{T_c} - 1$





$\delta F \sim \tau^2 \ln |\tau|$ $\qquad \delta C_v \sim -\ln|\tau|$

There is critical point, singularity in C_v. Onsager was the first who found phase-transition with Gibbs distribution.

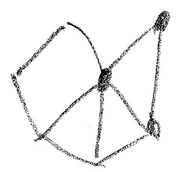
$\langle \sigma \rangle \sim (T_c - T)^{1/8}$ $\quad \beta = 1/8 \quad \gamma = 7/4 \quad \delta = 15$ 2D was found in 2D

Exact solution for 3D Ising model doesn't exist, but critical exponents are calculated: $\beta \simeq 1/3 \quad \gamma = 4/3 \quad \delta \simeq 5 \quad \alpha = 0$ 3D

For 4D Ising model Larkin Khmelnitski found that critical indeces of Ising model coincide with mean-field theory.

~ . ~ . ~ . ~ . ~ . ~ .

<u>M. Kac</u> realized what we should do with mean-field theory to receive the exact solution



$$J\sum_{\langle ij\rangle}\sigma_i\sigma_j \to \frac{J}{N}\underbrace{\sum_{ij}\sigma_i\sigma_j}_{\sim N^2}$$

$$\mathcal{H} = -\frac{J}{N}\left(\sum_i\sigma_i\right)^2 - h\sum_i\sigma_i$$

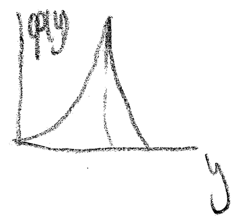
$$Z = \sum_{\sigma_i=\pm 1}\exp\left[\beta h\sum_i\sigma_i + \frac{\beta J}{N}\left(\sum_i\sigma_i\right)^2\right] \qquad e^{a^2} = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2-2xa}$$

$$\exp\frac{\beta J}{N}\left(\sum_i\sigma_i\right)^2 = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2-2x\sqrt{\frac{\beta J}{N}}\sum_i\sigma_i}$$

$$Z = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2}\left(\sum_{\sigma_i=\pm1}e^{\beta h\sum_i\sigma_i+2x\sqrt{\frac{\beta J}{N}}\sum_i\sigma_i}\right) =$$

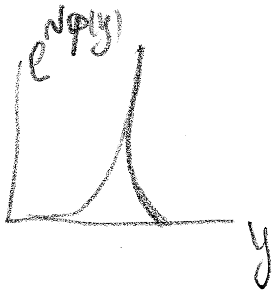
$$= \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}e^{-x^2}\left[2\cosh\left(\beta h+2x\sqrt{\frac{\beta J}{N}}\right)\right]^N = 2\sqrt{\frac{N}{\pi\beta J}}\int_{-\infty}^{\infty}dy\,e^{-\frac{Ny^2}{4\beta J}}\left[2\cosh\beta(h+y)\right]^N$$

$$Z \sim \int_{-\infty}^{\infty}dy\,e^{N\varphi(y)} \qquad \varphi(y) = -\frac{y^2}{4\beta J}+\ln\left(2\cosh\beta(h+y)\right)$$



$\varphi'(y) = 0$ $\quad \varphi''(y) < 0$ $\quad y = y_0$ maximum

in the limit of large N peak of $\varphi(y)$ will be sharper



only small part in the vicinity ~~near~~ peak will give contribution to integral

$$\boxed{\varphi(y) = \varphi(y_0) - \frac{1}{2}\varphi''(y_0)(y-y_0)^2}$$

saddle-point

Mean-field theory becomes exact only if you have long-range interaction. If ther largest ~~radius~~ of interaction r the system is r_0, mean-field will be valid for $r > r_0$, near critical point there is another length $\xi \to \infty$ $(T=0)$ and mean-field doesn't work. However bigger is r_0, the less interval where mean-field doesn't work. The shorter range of interaction r_0 the ~~bi~~worse is mean-field approximation

21.09.05

Lecture 3

Landau theory "Order parameter"

$T, V \quad F(T,V)$

$T, P \quad \boxed{G(P,T) = F + PV} \qquad \mu = \frac{1}{N} G(P,T) \qquad \boxed{\mu_1(P,T) = \mu_2(P,T)}$ equil. of phases

$d\mu = -\tilde{S}\,dT + \tilde{v}\,dp \qquad \tilde{S} = -\left(\frac{\partial \mu}{\partial T}\right)_p$ - entropy per particle

$\tilde{v} = +\left(\frac{\partial \mu}{\partial P}\right)_T$ - volume per particle

According to Erenfest classification:

1. First-order phase transitions: jump in first derivatives of thermodynamic potential $\Delta\tilde{v}, \Delta\tilde{S}$
2. Second-order phase transitions: jump in second derivatives... $\Delta\tilde{v} = 0 \quad \Delta\tilde{S} = 0 \qquad C_p = T\left(\frac{\partial S}{\partial T}\right)_p$

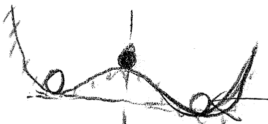
Landau introduced a general theory: any phase-transitions can be described by broken symmetry.

$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j \qquad J > 0$ - the system prefares all spins parallel but what about the direction

$\uparrow\uparrow\uparrow\uparrow \quad \downarrow\downarrow\downarrow\downarrow$

$\uparrow\uparrow\uparrow\uparrow$

The same is in mechanics



Symmetry of ordered phase is lower (cylindrical or circle) than symmetry of disordered phase (sphere)

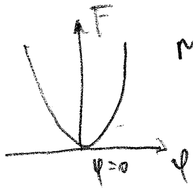
Near critical point of ferromag system has discrete (not continuous) broken symmetry: it was 4 equal directions, but the system has chosen only one ~~due to some~~ (the direction depends on some fluctuation.)

Order parameter: $\varphi = 0 \quad T > T_c$ more symmetric

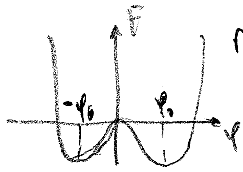
$\varphi \neq 0 \quad T < T_c$ less symmetric

$F(V,T) \qquad F = F_0 + \frac{a}{2}\varphi^2 + u\varphi^4 + \ldots$ for Hamiltonian spins $\uparrow$ and $\downarrow$ are equal (with the same E)

$u > 0$ (if φ is rather big F can't go to $-\infty$)



$r>0$



$r<0$ $\qquad r=\alpha(T-T_c)$

above critical point $\qquad$ below critical point

The ground state of the system $\frac{\partial F}{\partial \varphi}=0 \qquad \frac{\partial^2 F}{\partial \varphi^2}>0$

(1) $\boxed{\varphi=0}$ & $\boxed{\varphi^2=-\frac{r}{4u}}$ (2) $\qquad \frac{\partial^2 F}{\partial \varphi^2}=r+12u\varphi^2$

$r>0 \qquad r<0$

$$\varphi=\begin{cases}0, & T>T_c\\ \frac{\alpha(T_c-T)}{4u}, & T<T_c\end{cases} \qquad \beta=1/2 \qquad \varphi\sim\sqrt{T_c-T_q}$$

$$F=F_0+\frac{1}{2}r\left(-\frac{r}{4u}\right)+u\left(-\frac{r}{4u}\right)^2=F_0-\frac{\alpha^2(T_c-T)^2}{16u} \qquad T<T_c$$

$F=T_0 \qquad T>T_c$

$$S=-\frac{\partial F}{\partial T} \qquad S=\begin{cases}0, & T>T_c\\ -\frac{\alpha^2(T_c-T)}{8u}, & T<T_c\end{cases} \qquad \boxed{\begin{matrix}T=T_c\\ \Delta S=0\end{matrix}}$$

$$C=T\frac{\partial S}{\partial T} \qquad \boxed{\Delta C=\frac{\alpha^2 T_c}{8u}}$$

To find other critical indeces

$$F(h,T)=F_0+\frac{r}{2}\varphi^2+u\varphi^4-h\varphi$$

Minimization of free energy: $\frac{\partial F}{\partial \varphi}=0 \qquad r\varphi+4u\varphi^3-h=0$

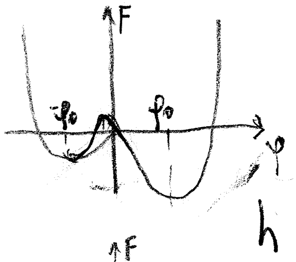
$$\frac{\partial^2 F}{\partial \varphi^2}>0 \qquad r+12u\varphi^2\geq 0$$

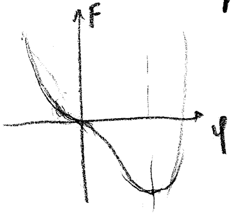
Interested in small $h\to 0$

(1) $T>T_c$: $\quad \varphi=\frac{h}{r}=\frac{h}{\alpha(T-T_c)} \to \gamma=1 \quad \chi=\left(\frac{\partial\varphi}{\partial h}\right)\Big|_{h=0} \quad \chi=\frac{1}{\alpha(T-T_c)}$

(2) $T<T_c$: $\varphi=\varphi_0+\delta\varphi \qquad \varphi_0=\sqrt{-\frac{r}{4u}} \qquad \delta\varphi=\frac{-h}{2r}$

$T=T_c \quad r=0 \quad 4u\varphi^3=h \qquad \varphi\sim h^{1/3} \quad \delta=3$





$$F = F_0 + \frac{r}{2}\varphi^2 + u\varphi^4 - h\varphi$$

with magnetic field first-order phase-trans

without magnetic field second-order phase-trans

$$F = F_0 + \frac{r}{2}\varphi^2 + u\varphi^4 + a\varphi^3 \quad \boxed{r\varphi + 4u\varphi^3 + 3a\varphi^2 = 0}$$

$r \to 0$ (1) $\varphi = 0$ (2) $r + 4u\varphi^2 + 3a\varphi = 0$

if $a\varphi^3$ is allowed is a first-order phase transition

else second-order (group of low-symmetry is subgroup of high-symmetry group)

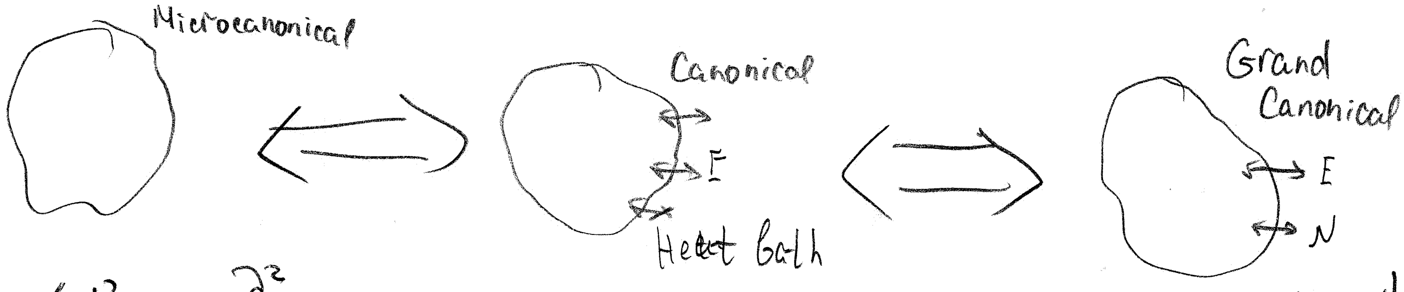
Only using Teylor theorem for F and coefficients = all Landau theory.
That is why all considered theories with long-range interactions can be described with Landau theory.

$$\langle H \rangle = \frac{\int dp\,dq\, H e^{-\beta H}}{\int dp\,dq\, e^{-\beta H}} = \frac{-\frac{\partial}{\partial\beta}\int dp\,dq\, e^{-\beta H}}{\int dp\,dq\, e^{-\beta H}} = -\frac{\partial}{\partial\beta}\ln Z = T^2\frac{\partial}{\partial T}\ln Z$$

$$\langle H^2 \rangle = \frac{-\frac{\partial^2}{\partial\beta^2} Z}{Z}$$

$$C_V = \frac{\overset{N}{\langle H^2\rangle} - \overset{N^2}{\langle H\rangle^2}}{T} = \frac{d}{dT}\langle H\rangle \sim N$$

$N \to \infty$



$H \Rightarrow H - \mu N$

$$\langle N^2 \rangle \leftrightarrow \frac{\partial^2}{\partial\mu^2} Z \qquad \langle N \rangle \leftrightarrow \frac{\partial}{\partial\mu} Z$$

$$\langle N^2 \rangle - \langle N \rangle^2 = T\left(\frac{\partial N}{\partial\mu}\right)_T$$

Grand canonical equals to canonical and microcanonical in thermodynamic limit if the fluctuations are small.

$$\left(\frac{\partial \mu}{\partial N}\right)_T = -\frac{V}{N}\left(\frac{\partial P}{\partial N}\right)_T = -\frac{V}{N}\left(\frac{\partial P}{\partial V}\right)_T \frac{\partial V}{\partial N} = -\left(\frac{V}{N}\right)^2 \left(\frac{\partial P}{\partial V}\right)_T$$

13

$$d\mu_T = -\frac{V}{N}\, dP \qquad \chi_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{V^2}{N^2}\,\frac{1}{V}\left(\frac{\partial N}{\partial \mu}\right)_T = \frac{V}{T}\left[\frac{\langle N^2\rangle}{\langle N\rangle^2} - 1\right]$$

The same is for magnetization $\chi = \left(\frac{\partial M}{\partial h}\right)_T \sim \frac{1}{T}\left[\frac{\langle M^2\rangle}{\langle M\rangle^2} - 1\right]$

Near critical point fluctuations are strong.

Mean-field theory doesn't take to account fluctuations.

29.09.05 | Lecture 4 Fluctuations and correlators (14)

compressibility $\varkappa \Leftrightarrow \frac{\langle N^2 \rangle}{\langle N \rangle^2} - 1$

susceptibility $\chi \Leftrightarrow \langle S^2 \rangle - \langle S \rangle^2$

As far as compressibility and susceptibility diverge at critical point $\Rightarrow$ fluctuations are enourmously big.

$h(\vec{x})$ exter — order $F[\varphi(\vec{x})]$

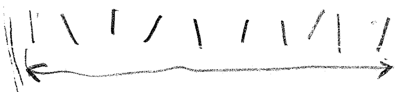
$\vec{\varphi}_i$ – order parameter $\quad \vec{\varphi}(\vec{r}) = \frac{\sum_{i \in A} \vec{\varphi}_i}{\text{Volume}_A}$

We'll assume that order parameter is smooth in considered volume

all typical inhomogeneties are larger than atomistic scales.

$$\boxed{F[\varphi(\vec{x})] = \int dV \Big[\underbrace{\frac{r}{2}\varphi^2(\vec{x}) + u\varphi^4(\vec{x})}_{\text{for homogeneous}} + \underbrace{\frac{c}{2}(\nabla\varphi(\vec{x}))^2}_{\text{for inhomogenity}} \Big]}$$

they are smooth



$L \gg r_0 \qquad \nabla \sim \frac{r_0}{L}$

function of function of infinite number of ~~[illegible]~~ multidimensional numbers $(x \ldots)$

<u>Ginsburg-Landau function</u>

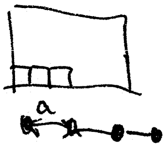
With external field: $F = F_{h=0} - \int dV \varphi(\vec{x}) h(\vec{x})$

Partition function: $\quad Z = \int \mathcal{D}\varphi(\vec{x}) \exp\{-\beta F[\varphi(\vec{x})]\}, \ \beta = \frac{1}{T}$

We can work with functional integral as with normal.

1). Our space is infinite  $\varphi(x+L, \ldots) = \varphi(x)$ periodic boundary condition

2). Space is periodic: „artificial" lattice



$a \to 0$

$L \to \infty$

Suppose I'm interested in average value of order parameter at $\vec{x}$

$$\langle\varphi(\vec{x})\rangle = \frac{\int D\varphi\, e^{-\beta F}\,\varphi(x)}{\int D\varphi\, e^{-\beta F}} \qquad \int D_\varphi\, e^{-\beta F|_{h=0} + \beta\int \varphi(x) h(x)\, dV}\,\varphi(\vec{x}) =$$

$$= \frac{1}{\beta}\frac{\delta}{\delta h(\vec{x})}\int D_\varphi\, e^{-\beta F|_{h=0} + \beta\int \varphi(\vec{x}) h(\vec{x})\, dV}$$

$$\langle\varphi(\vec{x})\rangle = \frac{1}{\beta}\frac{\delta}{\delta h(\vec{x})}\ln Z = \frac{1}{\beta}\frac{\delta Z/\delta h(\vec{x})}{Z}$$

$T > T_c \quad h \to 0$

$\langle\varphi(x)\rangle = \int dV' \chi(\vec{x},\vec{x}')h(\vec{x}')$

$$\langle\varphi(\vec{x})\varphi(\vec{x}')\rangle = \frac{1}{\beta^2}\frac{1}{Z}\frac{\delta^2 Z}{\delta h(\vec{x})\,\delta h(\vec{x}')}$$

theory of probability: dependent and independent events.

Fluctuations of order parameter are <u>not independent</u> at large distances. To describe this let us introduce correlator or correlation function:

$$\boxed{\Gamma(\vec{x},\vec{x}') = \langle\delta\varphi(\vec{x})\,\delta\varphi(\vec{x}')\rangle = \langle\varphi(\vec{x})\varphi(\vec{x}')\rangle - \langle\varphi(\vec{x})\rangle\langle\varphi(\vec{x}')\rangle}$$

$$\delta\varphi(\vec{x}) = \varphi(\vec{x}) - \langle\varphi(\vec{x})\rangle$$

$$\frac{\partial^2 \ln Z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\frac{1}{Z}\right) = \frac{1}{Z}\frac{\partial^2 Z}{\partial x\,\partial y} - \frac{1}{Z^2}\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial y}$$

$$\boxed{\Gamma(\vec{x},\vec{x}') = \frac{1}{\beta^2}\frac{\delta^2 \ln Z}{\delta h(\vec{x})\,\delta h(\vec{x}')}}$$

$\underline{\Gamma(|\vec{x}-\vec{x}'| \to \infty) = 0}$ at large distances correlations are wearer, at $\infty \to 0$.

$h=0 \quad \Gamma(\vec{x},\vec{x}') = \Gamma(\vec{x}-\vec{x}') = \underline{\underline{\Gamma(|\vec{x}-\vec{x}'|)}}$ for homogeneous

$T > T_c \quad \langle \varphi(\vec{x}) \rangle = 0 \quad h \to 0$

$\langle \varphi(x) \rangle = \frac{Num}{Den} \qquad Num = \int \mathcal{D}\varphi(\vec{x})\, \varphi(\vec{x})\, e^{-\beta F_{h=0} + \beta \int d\vec{x}'\, h(\vec{x}')\varphi(\vec{x}')} \approx$

$\approx \int \mathcal{D}\varphi(\vec{x})\varphi(\vec{x})\, e^{-\beta F|_{h=0}} \left[1 + \beta \int d\vec{x}'\, h(\vec{x}')\, \varphi(\vec{x}') \right] =$

$= \beta \int d\vec{x}'\, h(\vec{x}') \int \mathcal{D}\varphi\, \varphi(\vec{x})\, e^{-\beta F|_{h=0}}\, \varphi(\vec{x}')$

$Den = Z_{h=0} + O(h^2) \qquad \chi(\vec{x}, \vec{x}')\, \beta \langle \varphi(\vec{x}) \varphi(\vec{x}') \rangle$

$$\boxed{\delta \varphi(\vec{x}) = \int d\vec{x}'\, \chi(\vec{x}, \vec{x}')\, h(\vec{x}')}$$

$T < T_c \quad \chi(\vec{x}, \vec{x}') = \beta\, \Gamma(\vec{x}, \vec{x}')$

Without magnetic field corr. function can be written in terms of Fourier integral.

$$\boxed{\Gamma(\vec{x} - \vec{x}') = \int \frac{d^d q}{(2\pi)^d}\, \Gamma_{\vec{q}}\, e^{i\vec{q}(\vec{x}-\vec{x}')}} \qquad \underline{d=3} \quad \Gamma(\vec{q}) = \int d\vec{x}\, \Gamma(\vec{x})\, e^{-i\vec{q}\vec{x}}$$

$\Gamma = \Gamma(|\vec{x} - \vec{x}'|) \Longleftrightarrow \quad \Gamma_{\vec{q}} = \Gamma_{|\vec{q}|} \qquad q = |\vec{q}|$

Γ is periodic $\Rightarrow$ not all $\vec{q}$ is allowed, q is discrete

$e^{iq_x L_x} = 1 \qquad q_x = \frac{2\pi}{L_x} n_x \qquad n_x = 0, \pm 1, \pm 2 .$

$q_y = \frac{2\pi}{L_y} n_y, \quad q_z = \frac{2\pi}{L_z} n_z \qquad n_y = 0, \pm 1, \pm 2 ..;\ n_z = 0, \pm 1, \pm 2 ...$

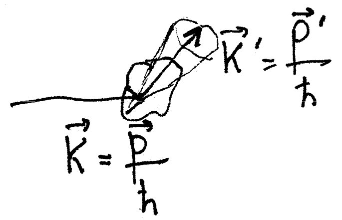
$$\boxed{\int \frac{d^d \vec{q}}{(2\pi)^d} \to \sum_{\vec{q}}}$$

Suppose we have external homogeneous field

$\langle \delta \varphi(x) \rangle = \langle \delta \varphi \rangle = \beta \int d\vec{x}'\, \Gamma(\vec{x} - \vec{x}')\, h \qquad \langle \delta\varphi \rangle = \chi h$

$$\boxed{\chi = \beta\, \Gamma_{\vec{q}=0}}$$

Fourier components of correlation functions can be measured in experiments, light scattering, neutron scattering, x-ray



$$\frac{d\sigma}{d\Omega_{\vec{x}'}} \sim |V_{\vec{k}-\vec{k}'}|^2 \qquad \overset{\text{potential}}{V_{\vec{q}}} = v_{\vec{q}}\, \delta\rho_{\vec{q}}$$

intensity of scattering $\quad \boxed{I_{\vec{k}\to\vec{k}'} \sim \Gamma_{\vec{k}-\vec{k}'}}$

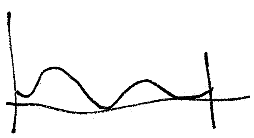
Spin-spin correlation function you can measure by neutron scattering (it has no charge, but magnetic moment) interacts with magn. moments..

Near critical points the scatterings is large due to enormous fluctuations. <u>Ornstein-Zernike</u>

liquid and gas are separated by <u>menisk</u>

at critical point the screen is black (broken symmetry)

$$\int \mathcal{D}\varphi(\vec{x}) \leftrightarrow \int \mathcal{D}\varphi_{\vec{q}} \qquad \varphi(\vec{x}) = \int \frac{d^d \vec{q}}{(2\pi)^d}\, \varphi_{\vec{q}}\, e^{i\vec{q}\vec{x}} = \sum_{\vec{q}} \varphi_{\vec{q}}\, e^{i\vec{q}\vec{x}}$$



$\varphi(\vec{x}) \leftrightarrow \varphi_{\vec{q}} \qquad \boxed{\varphi_{\vec{q}} = \varphi^*_{-\vec{q}}}$ for real functions.

$$\int dx_1 \ldots dx_n \to \int d^n x = \int d^n y \left| \frac{\partial(x_1, \ldots x_n)}{\partial(y_1, \ldots y_n)} \right|$$

$$x_i = x_i(y_n) \qquad \left| \frac{\partial(x_1, \ldots x_n)}{\partial(y_1, \ldots y_n)} \right| = \det \left\| \frac{\partial x_i}{\partial y_j} \right\| \quad N \times N$$

$$\frac{\delta\varphi(\vec{x})}{\delta\varphi_{\vec{q}}} \Rightarrow \frac{\partial\varphi(x_m)}{\partial\varphi_n} = e^{i\vec{q}_n \vec{x}_m}$$

we don't need it, because ~~det~~ Jacobian is just a number, that appears in Num and Den of corr. func.

Ornstein-Zernike $F = \frac{1}{2}\int d\vec{x}\,[c(\nabla\varphi)^2 + r\varphi^2]$ (18)

The only formal difference of Quantum field theory and Stat. mech of crit. phenomena is that field fluctuates both in space and time. We consider only space-fluct.

$$\boxed{\int d\vec{x}\;\varphi^2(\vec{x}) = \sum_{\vec{q}} |\varphi_{\vec{q}}|^2} \qquad F = \frac{1}{2}\sum_{\vec{q}} (cq^2 + r)|\varphi_{\vec{q}}|^2 \quad T > T_c$$

$$\boxed{\Gamma_{\vec{q}} = \langle |\varphi_{\vec{q}}|^2 \rangle} \qquad \Gamma_{\vec{q}} = \frac{\int D\varphi_{\vec{q}}\, |\varphi_{\vec{q}}|^2 \exp\{-\frac{1}{2T}\sum_{\vec{q}}(cq^2+r)|\varphi_{\vec{q}}|^2\}}{\int D\varphi_{\vec{q}} \exp\{-\frac{1}{2T}\sum_{\vec{q}}(cq^2+r)|\varphi_{\vec{q}}|^2\}}$$

It is Gaussian integral. The properties:

$$I(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$\langle x^2 \rangle = \frac{\int dx\, e^{-\alpha x^2} x^2}{\int dx\, e^{-\alpha x^2}} = \frac{-\frac{\partial}{\partial\alpha}\int dx\, e^{-\alpha x^2}}{\int dx\, e^{-\alpha x^2}} = -\frac{\partial}{\partial\alpha}\ln\int dx\, e^{-\alpha x^2} =$$

$$= -\frac{\partial}{\partial\alpha}\ln\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}$$

$$I = \int dx_1 \ldots dx_n \exp\{-\sum_{i,j=1}^{N} x_i A_{ij} y_j\}, \quad A_{ij} = A_{ji}, \quad x_i = \sum_j O_{ij} y_j$$

$$\det \| O_{ij} \| = 1$$

$$I = \int \prod_i dy_i \exp\{-\sum_i \lambda_i y_i^2\}, \quad \lambda_i > 0$$

$$I = \frac{\pi^{N/2}}{\sqrt{\lambda_1 \cdots \lambda_n}} \qquad \lambda_1 \cdots \lambda_n = \det \hat{A}$$

$$\vec{h}\vec{x} = \sum_i h_i x_i \qquad I[h_i] = \int dx_1 \ldots dx_n \exp[-\vec{x}\hat{A}\vec{x} + \vec{h}\vec{x}]$$

$$\boxed{\langle x_i x_j \rangle = \frac{\partial^2 \ln I}{\partial h_i \partial h_j} = \frac{1}{4}(\hat{A}^{-1})_{ij}} \qquad \vec{x} \to \vec{x} - \frac{1}{2}\hat{A}^{-1}\vec{h}$$

13.10.05 | Lecture 5 Fluctuations and correlations (19)

$$\langle A\rangle = \frac{\int \mathcal{D}\varphi(r)\exp\left[-\frac{F(\varphi)}{T}\right]A(\varphi)}{\exp\left[-\frac{F(\varphi)}{T}\right]} = \frac{\int \mathcal{D}\varphi_q e^{-F/T} A}{\int \mathcal{D}\varphi_q e^{-F/T}}$$

$$F = \frac{1}{2}\int d\vec{x}\,\left[c(\nabla\varphi)^2 + r\varphi^2\right] = \frac{1}{2}\sum_q (cq^2+r)|\varphi_q|^2$$

$$\sum_q \frac{d^dq}{(2\pi)^d}$$

Ornstein-Zernike

We want to calculate correlation functions

Our dimensionality of space is infinite.

$$\Gamma_q = \langle|\varphi_q|^2\rangle$$

$$Z[h_q] = \int \mathcal{D}\varphi_q \exp\left[-\frac{1}{2}\sum_q \alpha_q|\varphi_q|^2 - \sum_q \varphi_q h_{-q}\right] \ominus$$

$$\ominus \left\{\varphi_q = \varphi_q' - \frac{h_q}{\alpha_q}\right\} = \int \mathcal{D}\varphi_q' \exp\left\{-\frac{1}{2}\sum_q \alpha_q|\varphi_q'|^2 + \frac{1}{2}\sum_q \frac{|h_q|^2}{\alpha_q}\right\} =$$

$$= \boxed{Z[h=0]\exp\left\{\frac{1}{2}\sum_q \frac{|h_q|^2}{\alpha_q}\right\}}$$

$$h_{-q} = h_q^*$$
$$\varphi_{-q}^* = \varphi_q$$
$$\int d\vec{x}\,\varphi(x)h(x) = \sum_q \varphi_q h_q$$

it's exact expression for different h

$$\exp\left(-\sum_q \varphi_q h_{-q}\right) \simeq 1 - \sum_q \varphi_q h_{-q} + \frac{1}{2}\sum_{qq'}\varphi_q h_{-q}\varphi_{q'}h_{-q'} + \dots$$

$$Z[h_q] \simeq Z[h=0]\left\{1 - \sum_q\langle\varphi_q\rangle h_{-q} + \frac{1}{2}\sum_{qq'}\langle\varphi_q\varphi_{q'}\rangle h_{-q}h_{-q'} + \dots\right\}$$

$$Z[h_q] \simeq Z[h=0]\left\{1 + \frac{1}{2}\sum_q \frac{|h_q|^2}{\alpha_q} + \dots\right\}$$

$$-h_{-q}\int\mathcal{D}\varphi_q\,\varphi_q e^{-F/T} = -h_{-q}\langle\varphi_q\rangle Z[h]$$

$$\langle\varphi_q\rangle = 0 \qquad \langle\varphi_q\varphi_{q'}\rangle = 0 \quad -q \neq q'$$

$$\boxed{\langle|\varphi_q|^2\rangle = \frac{1}{\alpha_q}} \qquad \frac{1}{\alpha_q} = \frac{T}{r+cq^2} \qquad \Gamma_q = \frac{T}{r+cq^2}$$

$$r = \alpha(T-T_c) \qquad T \simeq T_c$$

$$\boxed{\Gamma_q = \frac{T_c}{\alpha(T-T_c)+cq^2}}$$

We consider that fluctuations are small

$$\varkappa^2 = \frac{\alpha(T-T_c)}{c} \longrightarrow \boxed{\Gamma_q = \frac{T_c}{c} \frac{1}{\varkappa^2+q^2}}$$

$$\Gamma_q(|\vec{x}|) = \int \frac{d^d q}{(2\pi)^d} \Gamma_q e^{i\vec{q}\vec{x}} = \frac{T_c}{c} \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\vec{x}}}{\varkappa^2+q^2}$$

$d=3$ $\quad \hat{z} \parallel \vec{x}$ $\quad \vec{q}\vec{x} = q|\vec{x}|\cos\theta$ $\quad t=\cos\theta$

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\vec{x}}}{\varkappa^2+q^2} = \int_0^\infty \frac{d^3q\, q^2}{(2\pi)^3(\varkappa^2+q^2)} \int_0^\pi d\theta \sin\theta\, e^{iq|\vec{x}|\cos\theta} = \int_0^\infty \frac{d^3q\, q^2}{(2\pi)^3(\varkappa^2+q^2)} \int_{-1}^1 dt\, e^{iq|x|t} =$$

$$= \int_0^\infty \frac{d^3q\, q^2}{(2\pi)^3(\varkappa^2+q^2)} \cdot \frac{\sin q|\vec{x}|}{q|\vec{x}|} = \frac{1}{4\pi} \frac{e^{-\varkappa|\vec{x}|}}{|\vec{x}|}$$

$$\xi = \frac{1}{\varkappa} = \sqrt{\frac{c}{\alpha(T-T_c)}} \qquad \boxed{\Gamma(|\vec{x}|) = \frac{T_c}{4\pi c} \frac{e^{-|\vec{x}|/\xi}}{|\vec{x}|}}$$

correlation length or radius

$\Gamma(|\vec{x}| \gg \xi) \approx 0$ order parameters at the distances large than ξ are not correlated

$$\boxed{\Gamma(|\vec{x}| \ll \xi) \simeq \frac{T_c}{4\pi c|\vec{x}|}} \longleftarrow \text{Ornstein-Zernike}$$

ξ is a typical length scale when ~~correlation~~ order parameter are correlated

$a \ll |\vec{x}| \ll \xi$ We can't use expression for the order ~~at~~ interatomic distances

$T \to T_c$ $\quad \xi \to \infty$

$$\Gamma(q) \sim \frac{1}{q^2+\varkappa^2} \qquad \boxed{\Gamma_q = \frac{1}{q^2} f(q\xi)} \qquad \frac{1}{q^2+\varkappa^2} = \frac{1}{q^2} \frac{1}{1+\frac{1}{q^2\xi^2}}$$

This expression ~~does~~ not always very accurate for experiments, Let's try to explain experiments in a better way.

$$\xi \sim \frac{1}{\sqrt{T-T_c}}$$

$$\Gamma_q = \frac{1}{q^{2-\eta}} f(q,\xi) \qquad \xi \sim \begin{cases} (T-T_c)^{-\nu} & T > T_c \\ (T_c - T)^{-\nu_1} & T < T_c \end{cases}$$

$q \ll \frac{1}{a}$

$T = T_c \quad \xi = \infty \quad \boxed{\nu' = \nu = 1/2} \quad \boxed{d=3}$

Ornstein-Zernike

$\boxed{\eta = 0}$

$\eta \simeq 0 \quad \nu \simeq \nu' \simeq 0.6 \div 0.7$ — experiments

$$\Gamma(|\vec{x}|) = \frac{1}{|\vec{x}|^{d-2+\eta}} \tilde{f}\left(\frac{|\vec{x}|}{\xi}\right) \qquad d=2 \quad \Gamma(|\vec{x}|) \sim \frac{1}{|\vec{x}|^{1/4}} \quad \eta = 1/4$$

$\nu = 1$

η was introduced by Fisher to explain 2D Ising model.

Calculations of fluctuations contribution of heat capacity.

$$Z = \int D\varphi_q \exp\left(-\frac{F}{T}\right) = \int D\varphi_q \exp\left(-\frac{1}{2}\sum_q \alpha_q |\varphi_q|^2\right) \qquad \alpha_q = \frac{cq^2 + r}{T}$$

$\mathcal{F} = -T \ln Z$ free energy for the chosen (given) configuration of order parameter (distribution)

$T \to T_c \quad \alpha \simeq \frac{c}{T_c}(q^2 + \varkappa^2)$

$$S = -\frac{\partial \mathcal{F}}{\partial T} \qquad C_v = -T\frac{\partial^2 \mathcal{F}}{\partial T^2}$$

$$\varphi_q = \varphi_{-q}^* \qquad Z = \frac{const}{\prod_q' \alpha_q} \qquad \mathcal{F} = T\sum_q{}' \ln \alpha_q = \frac{T}{2}\sum_q \ln \alpha_q + const$$

$\prod'$ means that we should integrate over half of space $\{\varphi_q = \varphi_{-q}^*\}$ over independent variables

in one half of plane φ are independent

q_x

$$\mathcal{F} = \frac{T}{2}\sum_{q<q_0} \ln(\alpha(T-T_c) + cq^2) + const'$$

logarithm is divergent, but our theory is restricted by interatomic distance $\to q < q_0$

$$C_v \sim T_c \frac{1}{(T-T_c) + cq^2}$$

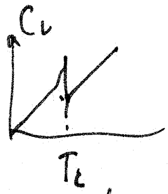
$$\Delta C_v^{\mathcal{H}} = \frac{\alpha^2}{2}\sum_q \frac{1}{[\alpha(T-T_c)+cq^2]^2} = \frac{\alpha^2}{2}\int_0^\infty \frac{dq\, q^2\, 4\pi}{(2\pi)^3}\frac{1}{[\alpha(T-T_c)+cq^2]^2}$$

$$\boxed{\Delta C_v^{\mathcal{H}} = \frac{\alpha^{3/2}}{16\pi c^{3/2}(T-T_c)^{1/2}}}$$

$T \to T_c$ divergence of heat capacity in Ornstein-Zernike

$$\boxed{\Delta C_v^{\text{meanfield}} = T_c\frac{\alpha^2}{8u}}$$

in Landau is finite jump



Meanfield gives good results for the system with long interaction. $\Delta C_v \sim \frac{1}{\sqrt{T-T_c}}$ is not small in the vicinity T_c and our assumption that contributions are small is wrong. It's due to big fluctuations, but we neglect φ^4 and were considering small fluctuations.

$$\Delta C_v^{\mathcal{H}} << \Delta C^{\text{mean-field}} \qquad 1 >> \frac{T-T_c}{T_c} >> \frac{u^2 T_c}{\alpha c^3}$$

$\alpha \sim 1$ $\quad u \sim E_{at}$ $\quad T_c \sim E_{at}$

$c(\nabla\varphi)^2 \sim E_{at}\left(\frac{r_0}{a}\right)^2$ $\qquad r_0$ - interatomic radius of interaction

$$\left(\frac{a}{r_0}\right)^6 << \left|\frac{T-T_c}{T_c}\right| << 1 \qquad \underline{\text{Ginsburg criterion}}$$

If you start with the system of small range interaction you can never use mean-field.

Mean-field theory $\rightarrow$ assume fluctuations $\rightarrow$ assume that corrections are small

Scaling concept Lecture 6

We considered the contribution of fluctuations of order parameter and we received:

I $\Delta C_v^{fl} \sim \frac{1}{|\tau|^{1/2}}$ $\quad \tau \to 0 \quad \tau = \frac{T-T_c}{T_c}$

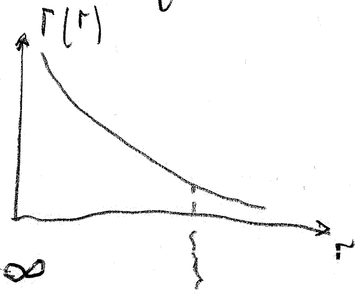
II $|\Delta C_v^{fl}| << |\Delta C_v^{m-f}| \Rightarrow \left(\frac{a}{r_0}\right)^6 << |\tau| << 1$ Ginsburg criterion

III $q << a^{-1}$ the contribution of fluctuations lead to the divergence of thermodyn. properties.

The problem is only in large-scale (small q) fluctuations

We'll consider long-waves fluctuations. $q \simeq \xi^{-1}$

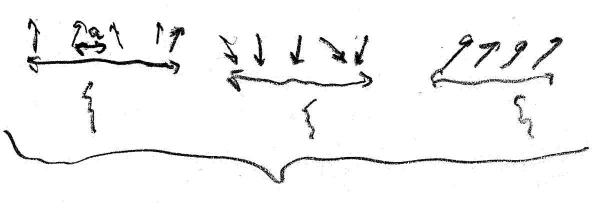
IV $\Gamma(r) \sim \frac{e^{-r/\xi}}{r}$



at T_c $\xi \sim |\tau|^{-\nu}$ very large ∞
and fluct. are correlated at large distances

Kadanoff (~1960) decimation (from Roman history) ~~where~~
the way to eliminate some irrelevant degrees of freedom
At high T: $T \gg T_c$ random fluctuations and it's much difficult to describe the behaviour all degrees of freedom are relevant. ↑ ↕ ↑ ↘ ↗ ↗ ↓ ↑ ↗

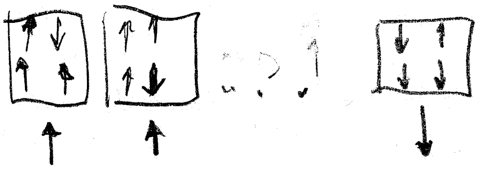
$T \simeq T_c$ $\quad \xi \to \infty$ mainly the orientation should be the same



Small-scale fluctuations within ξ:
1. they are small
2. they are irrelevant ~~inside~~ (a)

We can pass from the description of sites to the description of blocks $\rightarrow \uparrow \downarrow \nearrow$

The procedure: $a \to \lambda a$ (go from lattice with a to λa)



$\lambda > 1$

if the site of block is less than ξ

We pass to the block system: $\boxed{\uparrow\downarrow}\boxed{\uparrow\uparrow} \overset{?}{=} \Uparrow \quad \Downarrow$

$a \to \lambda a \to \lambda^2 a$ we can pass to superblocks 4×4

$a \to \lambda a \cancel{\lesssim} \xi$ close to T_c we can introduce blocks and consider interactions between blocks $\Rightarrow$ we kill a lot of degrees of freedom

This picture describes very well experiments, but there is no rigorous derivation.

How to formalize: the free energy of the block system should be the same as for initial system.

$\tau = \frac{T - T_c}{T_c}$ $\quad F(\tau, h)\dots$ other parameters are not important.

The form of free energy should be <u>similar</u>!?! for original and block system

$$F(\tau, h) = \lambda^{-d} F(\tilde{\tau}, \tilde{h})$$ d - dim. of space

<u>free energy per site</u>

$\boxed{\vdots\,\vdots}$ $\lambda \times \lambda \times \lambda \dots$

Similar doesn't mean identical parameters τ and h also can be renormalized

$$\left.\begin{aligned} \tilde{\tau} &= \lambda^y \tau \\ \tilde{h} &= \lambda^x h \\ \forall\, \lambda \end{aligned}\right\}$$ it's scaling hypothesis

It was justified by Wilson, proposed by Kadanoff

In terms of scaling hypothesis we'll have only 2 critical exponents instead of 7.

$$\boxed{F(\tau,h)=\lambda^{-d}F(\lambda^{y}\tau,\lambda^{x}h)}$$

$$\varphi=\frac{-\partial F}{\partial h}=-\lambda^{-d}\lambda^{x}\frac{\partial F(\lambda^{y}\tau,\lambda^{x}h)}{\partial F(\lambda^{x}h)}=$$

$$\boxed{\varphi=\lambda^{x-d}\varphi(\lambda^{y}\tau,\lambda^{x}h)}$$

$$\chi=\frac{\partial\varphi}{\partial h}$$

$$\boxed{\chi=\lambda^{2x-d}\chi(\lambda^{y}\tau,\lambda^{x}h)}$$

I'd establish the relationship between $\alpha,\beta,\gamma,\delta$ and x and y.

Let's consider scaling hypothesis for $h=0$

$$F(\tau)=\lambda^{-d}F(\lambda^{y}\tau) \qquad \lambda=\frac{1}{|\tau|^{1/y}}$$

$$F(\tau)=|\tau|^{\frac{d}{y}}\underbrace{F(\operatorname{sign}\tau)}_{\text{just a number}} \Longrightarrow 2-\alpha=\frac{d}{y}$$

$$F(\tau)\sim\begin{cases}(-\tau)^{2-\alpha'}, & \tau<0\\ (\tau)^{2-\alpha}, & \tau>0\end{cases}$$

$$\varphi(\tau,h)=\lambda^{x-d}\varphi(\lambda^{y}\tau) \qquad \lambda=\frac{1}{|\tau|^{1/y}},\ \tau<0 \Longrightarrow \beta=\frac{d-x}{y}$$

$$\varphi=|\tau|^{\frac{d-x}{y}}\varphi(-1) \qquad \varphi\sim(-\tau)^{\beta}$$

$$\chi\sim|\tau|^{\frac{d-2x}{y}} \qquad \chi\sim\begin{cases}\tau^{-\gamma}, & \tau>0\\ \tau^{-\gamma'}, & \tau<0\end{cases} \xRightarrow[\gamma=\gamma']{} \gamma=\frac{2x-d}{y}$$

At the critical point $\varphi\sim h^{1/\delta}$ $\tau=0$

$$\varphi(0,h)=\lambda^{x-d}F(0,\lambda^{x}h) \qquad \text{suitable choice } \lambda=\frac{1}{h^{1/x}}$$

$$\varphi(0,h)=h^{\frac{d-x}{x}}\varphi(0,1) \Longrightarrow \frac{d-x}{x}=\frac{1}{\delta}$$

Only two critical exponents x and y are independent.

Rushbrook inequality: $\alpha' + 2\beta + \gamma' \geq 2$

Scaling hypothesis: $\alpha' + 2\beta + \gamma' = 2 \qquad \alpha' = \alpha \quad \gamma' = \gamma$

Griffiths inequality: $\alpha' + \beta(1+\delta) \geq 2$

Scaling hypothesis: $\alpha' + \beta(1+\delta) = 2$

Let's γ and β be independent $\qquad \alpha' = \boxed{\alpha = 2 - \gamma - 2\beta}$

$$\beta(1+\delta) = 2 - \alpha' = \gamma + 2\beta \qquad 1 + \delta = \frac{\gamma + 2\beta}{\beta} \qquad \boxed{\delta = 1 + \frac{\gamma}{\beta}}$$

Let's check the results for mean-field theory.

Mean-field: $\alpha = \alpha' = 0 \quad \beta = \frac{1}{2} \quad \gamma = \gamma' = 1 \quad \delta = 3$

$$0 = 2 - 1 - 2\cdot\frac{1}{2} = 0 \qquad 3 = 1 + \frac{1}{1/2} = 3$$

2D Ising model (exact solution) $\alpha = \alpha' = 0 \quad \beta = \frac{1}{8} \quad \gamma = \gamma' = \frac{7}{4} \quad \delta = 15$

$$0 = 2 - \frac{7}{4} - 2\cdot\frac{1}{8} = 0 \qquad 15 = 1 + \frac{7}{4} \Big/ \frac{1}{8} = 15$$

Scaling hypothesis leads to the results that agree with both mean-field and Ising model. For all known results scaling hypothesis works.

We'll introduce scaling hypothesis for correlators.

The definition of correlators: $\Gamma(\vec{R}) = \langle \varphi(\vec{R})\, \varphi(0) \rangle \qquad R = |\vec{R}|$

$$T > T_c \qquad h = 0$$

$$\Gamma(R) = \langle \delta\varphi(\vec{R})\, \delta\varphi(0) \rangle \qquad \delta\varphi(\vec{R}) = \varphi(\vec{R}) - \varphi(0)$$

$\varphi \to \Phi \leftarrow$ order parameter for the blocks

$$\varphi(\tau, h) = L^{x-d} \underbrace{\Phi(L^y \tau, L^x \tau)}_{\text{order parameter of the block}}$$

I'll assume that ~~for~~ fluctuations is the same multiplyer (λ^{x-d}) and than I'll check. Scaling hyp. for fluctuat. is the same as for averages

Scaling hypothesis for correlators

$$\boxed{\Gamma(R,\tau,h) = \lambda^{2(x-d)}\Gamma\left(\frac{R}{\lambda}, \lambda^y\tau, \lambda^x h\right)} \qquad a \to \lambda a$$

I should measure all the distances in lattice constant a.
If we increase lattice constant by λ, all the distances should decrease!

$$\chi(\tau,h) = \frac{1}{T_c}\int d^dR\, \Gamma(R,\tau,h)$$

$$\underline{\chi(\tau,h)} = \frac{1}{T_c}\lambda^{2(x-d)}\int d^dR\, \Gamma\left(\frac{R}{\lambda}, \lambda^y\tau, \lambda^x h\right) = \left\{\vec{R}' = \frac{\vec{R}}{\lambda}\right\} =$$

$$= \frac{1}{T_c}\lambda^{2(x-d)}\lambda^d\int d^dR'\,\Gamma(R', \lambda^y\tau, \lambda^x h) = \lambda^{2x-d}\frac{1}{T_c}\int d^dR'\,\Gamma(R', \lambda^y\tau, \lambda^x h)$$

$$= \boxed{\lambda^{2x-d}\chi(\lambda^y\tau, \lambda^x h)}$$

$$h=0 \quad \lambda = \frac{1}{|\tau|^{1/y}} \qquad \Gamma(R,\tau) = \tau^{\frac{2(d-x)}{y}}\Gamma\left(R|\tau|^{1/y}, 1, \underbrace{|\tau|^{-\frac{x}{y}}h}_{0}\right)$$

$$R \sim \xi \qquad \Gamma \sim \varphi\left(\frac{R}{\xi}\right)$$

correlations should disappear

$$\frac{1}{\xi} \sim |\tau|^{1/y} \qquad \xi \sim \begin{cases} \tau^{-\nu}, & \tau > 0 \\ (-\tau)^{-\nu'}, & \tau < 0 \end{cases}$$

$$\Longrightarrow \nu' = \nu = \frac{1}{y}$$

Critical exponent η: $\Gamma(R, \tau=0, h=0) \sim \frac{1}{R^{d-2+\eta}}$

$$\Gamma(R) = \lambda^{-(d-2+\eta)}\Gamma\left(\frac{R}{\lambda}\right)$$

Scaling hypothesis for $\tau = 0$ $h=0$ $\Longrightarrow d-2+\eta = 2(d-x)$

Complete list of relations between critical exponents:

ν and η are independ.

$$2-\alpha = d\nu$$
$$\beta = \frac{d-2+\eta}{2}\nu$$
$$\gamma = (2-\eta)\nu$$
$$\delta = \frac{d+2+\eta}{d-2+\eta}$$

$\alpha' = \alpha$

$\gamma' = \gamma$

$\nu' = \nu$

This is result of scaling hypothesis just because we believe this hypothesis.

2D Ising model: $\nu = 1$ $\eta = \frac{1}{4}$ all relations are satisfied $\Rightarrow$ it works.

In mean-field theory there is no fluctuations and there is no η and ν ~~are not~~ but we can use Ornstein-Zernike + mean field theory

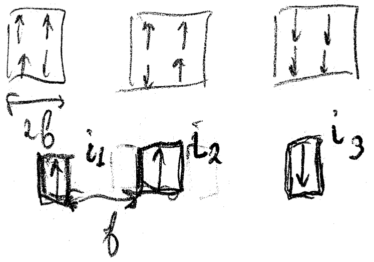
It works for $d=4$.

3.11.05

Renormalization group (RG)

Lecture 7

RG = coarse graining + scale transformation



1. Killing degrease of freedom
2. Rescaling the distances.

b - lattice constant

I. Coarse graining: $x_\beta \Rightarrow X_i$ gross variables $X_i = F_i(x_\beta)$

$$\mathcal{H}[x_\beta] \Rightarrow \mathcal{H}_{ef}[X_i]$$

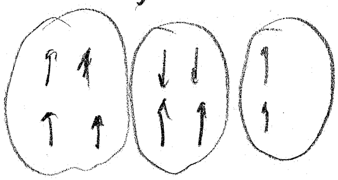
Suppose we want to describe Niagara fall:

1. Write Shroedinger eq., assuming everything is classic Newton for every molecule.

2. Introduce $\rho(\vec{r})$, $\vec{v}(\vec{r})$, $T(\vec{r})$ Navier-Stocks

In equillibrium statistical mechanics we should start with Gibbs distribution: $P \sim e^{-\beta \mathcal{H}[x_\mu]}$ $\quad \beta\mathcal{H} = \mathcal{H}$ $\quad P \sim e^{-\mathcal{H}[x_\mu]}$

$$Z = \int \prod_\mu dx_\mu \, e^{-\mathcal{H}[x_\mu]} =$$



$$= \int \prod_\mu dx_\mu \, e^{-\mathcal{H}[x_\mu]} \int \prod_i dX_i \, \delta(X_i - F_i(x_\mu)) = \int \prod_i dX_i \left(\int \prod_\mu dx_\mu e^{-\mathcal{H}[x_\mu]} \delta(X_i - F_i(x_\mu)) \right.$$

$$e^{-\mathcal{H}_{ef}[X_i]} = \int \prod_\mu dx_\mu \, e^{-\mathcal{H}[x_\mu]} \prod_i \delta(X_i - F_i(x_\mu))$$

$$Z = \int \prod_i dX_i \, e^{-\mathcal{H}_{ef}[X_i]}$$

An example how to introduce gross variables: (30)

For the system of harmonic oscillators:

$x_\mu = (p_\mu, q_\mu)$ $\quad \mathcal{H}[\{x_\mu\}] = \frac{1}{2}\sum_{\mu\nu} x_\mu A_{\mu\nu} x_\nu \quad \mathcal{H} = \frac{1}{2}\vec{x}^{T}\hat{A}\vec{x}$

$$\boxed{X_i = \sum_\mu f_{i\mu} x_\mu \qquad f_{i\mu} = \begin{cases} 1, & \mu \in i \\ 0, & \text{otherwise} \end{cases}}$$

you introduce just ~~the~~ the function to choose gross-variables

$$e^{-\mathcal{H}_{ef}[X_i]} = \int \prod_\mu dx_\mu\, e^{-\mathcal{H}[x_\mu]} \prod_i \delta\Big(X_i - \sum_\mu f_{i\mu} x_\mu\Big)$$

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}}$$

$$e^{-\mathcal{H}_{ef}} = \int \prod_\mu dx_\mu\, e^{-\mathcal{H}[x_\mu]} \prod_i \int \frac{dk_i}{2\pi}\, e^{ik_i X_i - ik_i \sum_\mu f_{i\mu} x_\mu} =$$

$$= \Big\{ g_\mu = \sum f_{i\mu} k_i \Big\} = \int \prod_i \frac{dk_i}{2\pi}\, e^{i\sum_i k_i X_i} \prod_\mu dx_\mu\, e^{-\frac{1}{2}\vec{x}\hat{A}x - i\vec{g}\vec{x}}$$

$$\int \prod_\mu dx_\mu \exp\Big\{-\frac{1}{2}\vec{x}\hat{A}\vec{x} - i\vec{g}\vec{x}\Big\} = \Big\{ \vec{x} = \vec{x}' + i\hat{A}^{-1}\vec{g} \Big\} = \cancel{\int \prod_\mu dx_\mu \exp\{-\tfrac{1}{2}x}$$

$$= \int \prod_\mu dx'_\mu \exp\Big\{-\frac{1}{2}\vec{x}'\hat{A}\vec{x}' - \frac{1}{2}\vec{g}\hat{A}^{-1}\vec{g}\Big\}$$

$$\boxed{\int \prod_\mu dx'_\mu \exp\Big(-\frac{1}{2}\vec{x}'\hat{A}\vec{x}'\Big) = \frac{(2\pi)^{N/2}}{(\det\hat{A})^{1/2}}}$$

$\hat{A}$ $N \times N$

$$e^{-\mathcal{H}_{ef}[X_i]} = \frac{1}{(2\pi)^{N/2}(\det\hat{A})^{1/2}} \int \prod_i dk_i \exp\Big\{ i\sum_i k_i X_i - \frac{1}{2}\sum_{ij\mu\nu} f_{i\mu} k_i (\hat{A}^{-1})_{\mu\nu} f_{j\nu} k_j \Big\}$$

$$\boxed{(\tilde{A}^{-1})_{ij} \stackrel{\text{def}}{=} \sum_{\mu\nu} f_{i\mu}(A^{-1})_{\mu\nu} f_{j\nu}}$$

$$e^{-\mathcal{H}_{ef}[X_i]} = \text{const}\, \exp\Big\{-\frac{1}{2}\sum_{ij} X_i \tilde{A}_{ij} X_j\Big\}$$

$$\boxed{\mathcal{H}_{ef}[X_i] = \frac{1}{2}\sum_{ij} X_i \tilde{A}_{ij} X_j + \text{const}}$$

the presentation of hamiltonian is not changed.

$A_{\mu\nu} \Rightarrow \tilde{A}_{ij}$

This trick is also important in crack propagation

You should introduce variables that is close to crack.

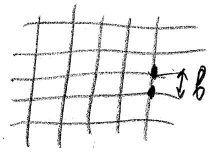
Multiscale simulations.

~~electric~~ elastic equations: Big pieces.

How the idea of coarse graining will work in theory of critical phenomena.

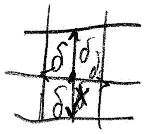
$$\mathcal{H} = \int d\vec{x} \left\{ \frac{1}{2} c (\nabla \varphi)^2 + \frac{1}{2} r \varphi^2 + u \varphi^4 \right\}$$ φ is continous

In real system we make discritization



$\varphi(\vec{x}) \to \varphi_{\vec{x}}$

$$\mathcal{H} = b^d \sum_{\vec{x}} \left[\frac{1}{2} r \varphi_{\vec{x}}^2 + u \varphi_{\vec{x}}^4 + \frac{1}{2} c \frac{1}{b^2} \sum_{\delta} (\varphi_{\vec{x}+\delta} - \varphi_{\vec{x}})^2 \right\}$$



$\int d\vec{x} = b^d \sum_{\vec{x}}$ d - space - dimensionality

$$CG: \varphi_{\vec{x}} \to \sum_{\vec{y} \in \vec{x}} \varphi_{\vec{y}} \qquad b \to \mathcal{L} b$$

 $\mathcal{L} = 2$

$$\begin{cases} \varphi_{\vec{x}} \to \varphi_{\vec{x}'} \\ b \to \mathcal{L} b \end{cases}$$

$$\boxed{\lambda_{\mathcal{L}} \varphi_{\vec{x}}' = \mathcal{L}^{-d} \sum_{\vec{y} \in \vec{x}} \varphi_{\vec{y}}}$$

hypothesis that our hamiltonian will have the same form

$$\mathcal{H}'[\varphi'] = b^d \sum_{\vec{x}'} \left\{ \frac{1}{2} c' b^{-2} \sum_{\delta} (\varphi_{\vec{x}'}' - \varphi_{\vec{x}+\delta}')^2 + \frac{1}{2} r' \varphi_{\vec{x}'}' + u \varphi_{\vec{x}'}'^4 \right\}$$

The hypothesis is that some terms are irrelevant.

Relevant terms will reproduce the same shape of hamilt.

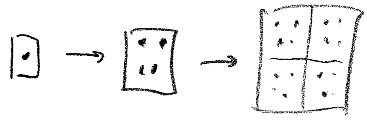
$\vec{\mu} = (r, u, c) \Rightarrow \vec{\mu}' = (r', u', c')$ new terms may appear $\varphi^2(\nabla\varphi)^2 + \varphi^6 + \varphi^8 \ldots$

But all new parameters are function of old

$$\vec{\mu}' = \vec{\mu}'[\vec{\mu}]$$

$\hat{R}_\lambda$ $\quad b \to \lambda b$, $\quad \varphi \to \varphi'$

Assumption that this renormaliz transformation satisfies group aciom: $\boxed{\hat{R}_\lambda \hat{R}_{\lambda'} = \hat{R}_{\lambda\lambda'}}$



the provement that close to critical point there is no length scale except ξ but ξ is very large. $b \to 4a$, $b \to 2a \to 2a$

Another assumption

Fixed point

$$\vec{\mu}^* = \hat{R}_\lambda \vec{\mu}^*$$

$$\boxed{\vec{\mu}^* = \hat{R}_\lambda \vec{\mu}^*, \quad T = T_c}$$

mathematical critical point corresponds to fixed point

The change of the space really means nothing at critical point ($\xi \to \infty$)

Near critical point: $\vec{\mu} = \vec{\mu}^* + \delta\vec{\mu}$ $\qquad \delta\vec{\mu}' = \hat{R}_\lambda \delta\vec{\mu}$

$$(\hat{R}_\lambda)_{\alpha\beta} = \left(\frac{\partial \mu'_\alpha}{\partial \mu_\beta}\right)_{\vec{\mu} = \vec{\mu}^*} \qquad \hat{R}_\lambda \hat{R}_{\lambda'} = \hat{R}_{\lambda\lambda'} \qquad [\hat{R}_\lambda, \hat{R}_{\lambda'}] = 0$$

$$[A, B] = 0 \implies \boxed{\begin{matrix} \hat{A}\vec{e} = a\vec{e} \\ \hat{B}\vec{e} = b\vec{e} \end{matrix}}$$

if matrices are commute than they have the same eigenstates

$$\forall \lambda: \hat{R}_\lambda \vec{e}_j = \rho_j(\lambda)\vec{e}_j$$

$$\rho_j(\lambda)\rho_j(\lambda') = \rho_j(\lambda\lambda') \qquad \rho_j(\lambda) = \lambda^{y_j}$$

$$\delta\vec{\mu} = \sum_j t_j \vec{e}_j \qquad \delta\vec{\mu}' = \sum_j t'_j \vec{e}_j \qquad \boxed{t'_j = t_j \lambda^{y_j}}$$

We are interested what happens when $\lambda \to \infty$

$$\delta\vec{\mu}' = \hat{R}\,\delta\vec{\mu} = \sum_j t_j \hat{R}\vec{e}_j = \sum_j t_j \rho_j \vec{e}_j = \sum_j t_j \lambda^{y_j} \vec{e}_j$$

$t_1(T)$ $\qquad \vec{\mu}' = \vec{\mu}^* + t_1(T)\lambda^{y_1}\vec{e}_1 + t_2(T)\lambda^{y_2}\vec{e}_2 + \ldots$ $\qquad y_1 > y_2 > y_3 \ldots$

$\lambda \gg 1$ $\qquad \vec{\mu}' = \vec{\mu}^* + t_1(T)\lambda^{y_1}\vec{e}_1 + \ldots$ $\qquad \lambda \to \infty$, $\quad y_1 > 0$

$t_j(T) = A(T-T_c)$ $\qquad \vec{\mu}(T_c) = \vec{\mu}^* \quad t_j(T_c) = 0$ 33

$$\vec{\mu}' \simeq \vec{\mu}^* + A(T-T_c)\, \lambda^{y_1} \vec{e}_1 + \ldots = \vec{\mu}^* \pm \left(\frac{\lambda}{\xi}\right)^{y_1} \vec{e}_1 + \ldots$$

$$\nu = \frac{1}{y} \qquad \xi = |A(T-T_c)|^{-\nu}$$

$$\lambda_\lambda \varphi_{\vec{x}} = \lambda^{-1} \sum_{\vec{y} \in \vec{x}} \varphi'_{\vec{y}} \qquad \boxed{\lambda_\lambda = \lambda^a}$$

$$\langle \varphi_{\vec{x}} \varphi_{\vec{x}+\vec{R}} \rangle = \lambda^{2a} \langle \varphi_{\vec{x}/\lambda} \varphi_{\frac{\vec{x}+R}{\lambda}} \rangle$$

$$\boxed{T = T_c \qquad \langle \varphi_{\vec{x}} \varphi_{\vec{x}+R} \rangle \sim \frac{1}{R^{d-2+\eta}}}$$

$$\boxed{a = -\frac{d-2+\eta}{2}}$$

it is an only point where critical point exist $\qquad \boxed{\nu = \frac{1}{y_1}}$

To justify (1) (c, r, u) are relevant parameters $\cancel{\varphi^6}$

(2) Existance of $\mu^* \Leftrightarrow a \leftrightarrow y$

(3) $t_i \sim T - T_c$, $y_i > 0 \Leftrightarrow \nu = \frac{1}{y_i}$

10.11.05

Renormalization group

Lecture 8

We are interested in particular system

$$\mathcal{H} = \frac{1}{2}\int d\vec{x}\,\left[r\,\varphi^2(\vec{x}) + c\left(\nabla\varphi(\vec{x})^2\right)\right] + u\int d\vec{x}\,\varphi^4(\vec{x})$$

In Ma book: $\tau = u_2$ $u = \frac{u_4}{8}$ $\vec{\varphi} = (\varphi_1, \dots \varphi_n)$ $n=3$ we consider $n=1$

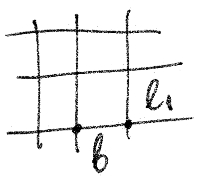
$$\mathcal{H} = \frac{1}{2}\sum_q (c + rq^2)|\varphi_q|^2 + uL^{-d}\sum_{qq'q''} \varphi_{\vec{q}'}\varphi_{q''}\varphi_{\vec{q}}\,\varphi_{-\vec{q}-\vec{q}'-\vec{q}''}$$

$$\boxed{\begin{aligned} &\sum_q = L^d\int\frac{d^d\vec{q}}{(2\pi)^d} \\ &\int d\vec{x}\,\varphi^2(x) = \sum_q |\varphi_q|^2 \\ &\int d\vec{x}\,(\nabla\varphi(\vec{x}))^2 = \sum_q q^2|\varphi_q|^2 \end{aligned}}$$

Some integrals are divergent.

1. $q \to 0$ $\Delta C_{fe}^{v} \sim \frac{1}{|\tau|^{1/2}}$ $\tau = 0$ $q \sim \frac{1}{\xi}$ are relevant physics

2. $q \to \infty$



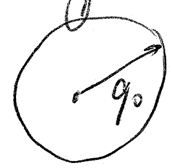
Lattice $\varphi(\vec{x}) = \varphi(\vec{x} + b\vec{e}_1)$

$$q_{max} = \frac{2\pi}{b}$$

$q \to \infty$ corresponds $x \to 0$, but physically $|x| \gtrsim b$ for critical behaviour is completely irrelevant what happens at small distances.

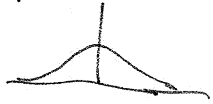
Nothing is dependent on large q-region.

All integrals are restricted $\displaystyle\sum_q = L^d \int_{|\vec{q}|<q_0} \frac{d^d q}{(2\pi)^d}$



Nothing is dependent on q_0: Rescaling $q_0 \to \frac{q_0}{\lambda}$ $\lambda > 1$ nothing should happen. Cut-off in q-space is arbitrary.

$\boxed{\text{Gaussian case } u=0:}$ $e^{-x^2/2\sigma^2}$



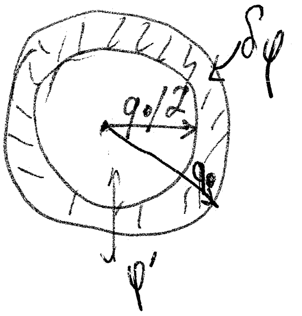
$$Z = \int\prod_q d\varphi_q\, e^{-\mathcal{H}}$$

Free-field, non-interactive field

Rescaling: I $\varphi(\vec{x}) = \varphi'(x) + \delta\varphi(x)$

$$\varphi'(\vec{x}) = \sum_{|\vec{q}|<\frac{q_0}{2}} \varphi_{\vec{q}}\, e^{i\vec{q}\vec{x}} \qquad \delta\varphi(\vec{x}) = \sum_{\frac{q_0}{2}<|\vec{q}|<q_0} \varphi_{\vec{q}}\, e^{i\vec{q}\vec{x}}$$

is irrelevant



Coarse-graining in q-space is almost trivial.

$$Z = \int \prod_{q<q_0} d\varphi_q\, e^{-\mathcal{H}} = \int \prod_{q<\frac{q_0}{2}} d\varphi_q' \prod_{\frac{q_0}{2}<q<q_0} d(\delta\varphi_q) \exp\left\{-\frac{1}{2}\sum_{q<\frac{q_0}{2}} (cq^2+r)|\varphi_q|^2\right\}$$

$$\cdot \exp\left\{-\frac{1}{2}\sum_{\frac{q_0}{2}<q<q_0} (cq^2+r)|\varphi_q|^2\right\}$$

$$= \text{const} \int \prod d\varphi_q' \exp\left\{-\frac{1}{2}\sum_{q<\frac{q_0}{2}} (cq^2+r)|\varphi_q|^2\right\}$$

$$\text{const} = \int \prod d(\delta\varphi_q) \exp\left\{-\frac{1}{2}\sum_{\frac{q_0}{2}<q<q_0} (cq^2+r)|\delta\varphi_q|^2\right\}$$

$$\boxed{\delta\varphi_q = \begin{cases} 0, & q<\frac{q_0}{2} \\ \varphi_q, & \text{otherwise} \end{cases}}$$

(I) $\mathcal{H} \to \frac{1}{2}\sum_{|q|<\frac{q_0}{2}} (cq^2+r)|\varphi_q|^2$

Skip ~~all~~ degrees of freedom with $\frac{q_0}{2}<q<q_0$

$x \to 2x$ $\quad q = q'/2 \quad 0<|q'|<q_0$

$q \to q/2$

$$\mathcal{H} = \frac{1}{2} 2^{-d} \sum_{|\vec{q}'|<q_0} (c\,2^{-2} q'^2 + r)|\varphi_{q'}|^2 = [\varphi_{q'} \to 2^{1-\eta/2}\varphi_q] =$$

$$= \frac{1}{2} 2^{-d} \sum_{|\vec{q}|<q_0} (c\,2^{-\eta} q^2 + r\,2^{2-\eta})|\varphi_q|^2$$

$c' = c\,2^{-\eta} \qquad r' = r\,2^{2-\eta}$

$\mathcal{H} \to 2^{-d}\mathcal{H}$

if we choose $\eta = 0$ $c' = c$ $r' = r Z^2$

$y_1 = 2 \Rightarrow \nu = \frac{1}{2}$ when there is no interaction between transformation

It was Gaussian case.

What happens if we introduce term $u\varphi^4$

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ $\qquad \mathcal{H}_0 = \frac{1}{2} \sum\limits_{|\vec{q}|<q_0} (cq^2 + r)|\varphi_q|^2$

$$\mathcal{H}_1 = u \int d\vec{x}\, \varphi^4(\vec{x}) = uL^d \sum_{|\vec{q}|,\vec{q}',\vec{q}''|<q} \varphi_{\vec{q}} \varphi_{\vec{q}'} \varphi_{\vec{q}''} \varphi_{-\vec{q}-\vec{q}'-\vec{q}''}$$

$u \to 0$ $\quad u \neq 0$

Gaussian case. $u = 0$, $\eta = 0$, $\nu = \frac{1}{2}$ OZ theory

Question: Effect of small u: $\begin{cases} d > 4 \\ d < 4 \end{cases}$

$$Z = \int \mathcal{D}\varphi\, e^{-\mathcal{H}_0 - \mathcal{H}_1} = \int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{H}_1^n = Z_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \mathcal{H}_1^n \rangle$$

$$Z_0 = \int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} \qquad \langle A \rangle = \frac{\int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} A}{\int \mathcal{D}\varphi\, e^{-\mathcal{H}_0}}$$

Cumulants $\quad \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{H}_1^n \overset{def}{=} \exp\left[+ \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle \mathcal{H}_1^n \rangle_c \right]$

$$e^A = \sum_{n=0}^{\infty} \frac{A^m}{m!}$$

$\langle \mathcal{H}_1 \rangle_c = \langle \mathcal{H}_1 \rangle$

$\langle \mathcal{H}_1^2 \rangle_c = \langle \mathcal{H}_1^2 \rangle - \langle \mathcal{H}_1 \rangle^2 = \langle (\mathcal{H}_1 - \langle \mathcal{H}_1 \rangle)^2 \rangle$

$\langle \mathcal{H}_1^3 \rangle_c = \langle \mathcal{H}_1^3 \rangle - 3 \langle \mathcal{H}_1^2 \rangle \langle \mathcal{H}_1 \rangle$

$$\mathcal{Z} = \int \mathcal{D}\varphi\, e^{-(\mathcal{H}_0 + \mathcal{H}_1)} = \int \mathcal{D}\varphi \exp\left[-\mathcal{H}_0 - \langle \mathcal{H}_1 \rangle_c + \frac{1}{2}\langle \mathcal{H}_1^2 \rangle_c - \frac{1}{3!}\langle \mathcal{H}_1^3 \rangle_c \ldots\right]$$

$$\mathcal{Z} = \int \mathcal{D}\varphi\, e^{-\mathcal{H}_{ef}} \qquad \mathcal{H}_{ef} = \mathcal{H}_0 + \langle \mathcal{H}_1 \rangle_c - \frac{1}{2}\langle \mathcal{H}_1^2 \rangle_c \ldots$$

$$\langle \mathcal{H}_1 \rangle = \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} u \varphi^4(\vec{x}) = \cancel{\frac{u}{\mathcal{Z}_0} L^{-d} \sum_{\vec{q},\vec{q}',\vec{q}''} \int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} \langle}$$

$$= \frac{u}{\mathcal{Z}_0} L^{-d} \sum_{\vec{q},\vec{q}',\vec{q}''} \mathcal{D}\varphi\, e^{-\mathcal{H}_0} \varphi_{\vec{q}} \varphi_{\vec{q}'} \varphi_{\vec{q}''} \varphi_{-\vec{q}-\vec{q}'-\vec{q}''}$$

<u>Wick theorem</u>

$$\vec{\varphi} = (\varphi_1, \varphi_2 \ldots \varphi_n)$$

$$\mathcal{H}_0 = \frac{1}{2}\sum_{i,j=1}^{N} \varphi_i A_{ij} \varphi_j = \frac{1}{2} \vec{\varphi}^T \hat{A} \varphi = \frac{1}{2}(\vec{\varphi}, \hat{A}\vec{\varphi})$$

$$\mathcal{Z}_0 = \int \mathcal{D}\varphi \exp(-\mathcal{H}_0) = \frac{(2\pi)^{N/2}}{(\det \hat{A})^{1/2}}$$

$$\langle \varphi_1 \varphi_2 \ldots \varphi_N \rangle = \frac{\int \mathcal{D}\varphi\, e^{-\mathcal{H}_0} \varphi_1 \ldots \varphi_n}{\int \mathcal{D}\varphi\, e^{-\mathcal{H}_0}}$$

Generating functional (we prefere to work with exp)

$$\mathcal{Z}[\vec{h}] = \int \mathcal{D}\varphi \exp\left[-\frac{1}{2}(\vec{\varphi}, \hat{A}\vec{\varphi}) + (\vec{h}, \vec{\varphi})\right] = \int \cancel{\prod_{i=1}^{N} d\varphi_i \exp\left[-\frac{1}{2}\right.}$$

$$= \int \prod_{i=1}^{N} d\varphi_i \exp\left[-\frac{1}{2}\sum_{i,j=1}^{N} \varphi_i A_{ij} \varphi_j + \sum_{i=1} h_i \varphi_i\right]$$

$$\frac{\partial^n \mathcal{Z}}{\partial h_1 \ldots \partial h_n} = \int \prod_{i=1}^{N} d\varphi_i \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} \varphi_i A_{ij} \varphi_j\right) \varphi_1 \ldots \varphi_n \exp\left(\sum_{j=1}^{N} h_j \varphi_j\right) =$$

$$= \mathcal{Z} \langle \varphi_1 \varphi_2 \ldots \varphi_n \rangle \quad \vec{h} = 0$$

$$Z[\vec{h}] = [\vec{\varphi} = \vec{\varphi}' - \hat{A}^{-1}\vec{h}] = Z_0 \exp\left\{\frac{1}{2}(\vec{h}, \hat{A}^{-1}\vec{h})\right\} =$$

$$= Z_0 \exp\left[\sum_{i=1}^{N} h_i \langle\varphi_i\rangle_c + \frac{1}{2}\sum_{ij} h_i h_j \langle\varphi_i \varphi_j\rangle_c + \frac{1}{3!}\sum_{ijk} h_i h_j h_k \langle\varphi_i \varphi_j \varphi_k\rangle_c \ldots\right]$$

$$\langle\varphi_i\rangle_c \stackrel{def}{=} \langle\varphi_i\rangle$$

$$\langle\varphi_i \varphi_j\rangle_c \stackrel{def}{=} \langle\varphi_i \varphi_j\rangle - \langle\varphi_i\rangle\langle\varphi_j\rangle$$

$$\langle\varphi_i \varphi_j \varphi_k\rangle \stackrel{def}{=} \langle\varphi_i \varphi_j \varphi_k\rangle - \langle\varphi_i \varphi_j\rangle\langle\varphi_k\rangle - \langle\varphi_j \varphi_k\rangle\langle\varphi_i\rangle - \langle\varphi_i \varphi_k\rangle\langle\varphi_j\rangle$$

Wick theorem: $\langle\varphi_i\rangle = 0$ $\quad \langle\varphi_i \varphi_j\rangle = A^{-1}_{ij}$ $\quad$ || All $\langle\varphi_1 \ldots \varphi_n\rangle = 0$, $n \geq 3$

$\langle\varphi_1\rangle = 0 \quad \langle\varphi_1 \varphi_2 \varphi_3\rangle = 0 \quad \langle\varphi_1 \ldots \varphi_{2n+1}\rangle = 0$ odd terms $= 0$

Even-order terms: $\langle\varphi_1 \varphi_2\rangle = (\hat{A}^{-1})_{12}$

$$\langle\varphi_1 \varphi_2 \varphi_3 \varphi_4\rangle = \langle\varphi_1 \varphi_2\rangle\langle\varphi_3 \varphi_4\rangle + \langle\varphi_1 \varphi_3\rangle\langle\varphi_2 \varphi_4\rangle + \langle\varphi_1 \varphi_4\rangle\langle\varphi_2 \varphi_3\rangle$$

$$\langle\varphi_1 \varphi_2 \varphi_3 \varphi_4 \varphi_5 \varphi_6\rangle = \ldots \qquad \boxed{\mathcal{H} = \frac{1}{2}(\vec{\varphi}, \hat{A}\vec{\varphi})} \Rightarrow$$ Feynman diagrams

Feynman diagrams is a way of presentations of Wick theorem.

17.11.05

Lecture 9

$$\mathcal{H} = \frac{1}{2}\int d^dx\,[c(\nabla\varphi)^2 + r\varphi^2] + u\int d^dx\ \varphi^4(x)$$ Ginsburg-Landau Hamiltonian

I $\varphi(x) = \sum_{|q|<q_0} \varphi_q e^{i\vec{q}\vec{x}}$ q_0 - cutoff wave vector

II $\varphi = \varphi' + \delta\varphi$ Long-wave part: $\varphi'(\vec{x}) = \sum_{|q|<q_0/\lambda} \varphi_q e^{i\vec{q}\vec{x}}$

Short-wave part: $\delta\varphi = \sum_{\frac{q_0}{\lambda}<|q|<q_0} \varphi_q e^{i\vec{q}\vec{x}}$

III $e^{-\mathcal{H}} \longrightarrow \int \mathcal{D}(\delta\varphi) e^{-\mathcal{H}} = e^{-\mathcal{H}'}$ $\mathcal{H}'$ for long wavelength fluctuat.

Last time: $u=0$

Today: $u \neq 0$, $u \to 0$. First order in u.

$u=0$: $\mathcal{H}_0 = \frac{1}{2}\int d^dx\, c(\nabla\varphi)^2 + \frac{1}{2} r \int d^dx\, \varphi^2$

$\boxed{\vec{x} \to \vec{x}' = \vec{x}/\lambda}$ $\boxed{\varphi(\vec{x}) \to \varphi'(\vec{x}')\lambda^{1-d/2}}$

$c\int d^dx(\nabla\varphi)^2 \longrightarrow c'\int d^dx'(\nabla'\varphi')^2$

$\boxed{c=c'}$ grad term is invariant

$\boxed{r \to r' = r\lambda^2}$ $\eta = 0$ $\nu = \frac{1}{2}$

Put $\mathcal{H}_1 = u\int d^dx\ \varphi^4(x)$ Calculate: $\int \mathcal{D}(\delta\varphi) e^{-\mathcal{H}_0 - \mathcal{H}_1} = ?$

Preliminary: $\mathcal{H}_1 \to \lambda^d(\lambda^{1-d/2})^4$. $\mathcal{H}_1' = \lambda^{4-d}\mathcal{H}_1$

Hypothesis: $d>4$ Critical point is described by Gaussian (OZ) theory

$$u' = u\,l^{4-d} \xrightarrow[l\to\infty]{} 0$$

$d<4$ $\quad u' \to \infty$ " Gaussian point is unstable "?!

<u>Another way</u>: Fluctuation heat capacity: $\Delta C^{fl} \sim \frac{1}{|\tau|^{1/2}} \xrightarrow[\tau\to 0]{} \infty$ (3D)

Arbitrary d: $\Delta C^{fl} \sim |\tau|^{d/2-2}$ $\quad d>4$

ΔC_{fl} is always smaller than $\Delta C^{fl}_{\text{Mean-field}}$

$$e^{-\mathcal{H}_{ef}} = \frac{\int \mathcal{D}(\delta\varphi)\, e^{-(\mathcal{H}_0+\mathcal{H}_1)}}{\int \mathcal{D}(\delta\varphi)\, e^{-\mathcal{H}_0}} = e^{-\mathcal{H}_0 - \langle \mathcal{H}_1\rangle - \frac{1}{2}(\langle \mathcal{H}_1^2\rangle - \langle \mathcal{H}_1\rangle^2)}$$

$$\mathcal{H}_{ef} = \mathcal{H}_0 + \langle \mathcal{H}_1 \rangle \qquad \langle \mathcal{H}_1 \rangle = \frac{u \int \mathcal{D}(\delta\varphi)\, e^{-\mathcal{H}_0}\left[\int d^d x\, \varphi^4(x)\right]}{\int \mathcal{D}(\delta\varphi)\, e^{-\mathcal{H}_0}} =$$

$$\langle \varphi^4(x)\rangle = \langle \varphi\,\varphi\,\varphi\,\varphi \rangle = 3\langle \varphi^2(x)\rangle^2$$

$$= u \int d^4 x\, \langle \varphi^4(x) \rangle$$

To calculate $\langle \varphi^2(x)\rangle = \langle (\varphi' + \delta\varphi)^2\rangle = \varphi'^2 + \langle (\delta\varphi)^2\rangle$

Averaging over $\underline{\delta\varphi}$ at fixed φ' !!! (short-wave)

To calculate $\langle (\delta\varphi)^2\rangle$: $\langle \varphi^2\rangle = \sum_{|q|<q_0} \langle |\varphi_q|^2\rangle$ which means that

$$\langle (\delta\varphi)^2 \rangle = \sum_{\frac{q_0}{l} < |q| < q_0} \langle |\varphi_q|^2 \rangle$$

At the critical point $T = T_c$: $\mathcal{H}_0 = \frac{1}{2} \sum_{|q|<q_0} cq^2 |\varphi_q|^2$

$\underline{\underline{r=0}}$ $\quad \langle |\varphi_q|^2 \rangle = \frac{1}{cq^2}$ at $T = T_c$

We need to calculate: $\langle (\delta\varphi)^2 \rangle = \sum_{\frac{q_0}{2}<|q|<q_0} \frac{1}{cq^2}$

3D $\quad \int d^3x\, f(|\vec{x}|) = \int_0^\infty dx\, 4\pi x^2 f(x)$

0

$$\int d^d x\, f(|\vec{x}|) = \int_0^\infty dx\, S_d\, x^{d-1} f(x)$$

useful trick: $f(|\vec{x}|) = e^{-x^2} = e^{-x_1^2 - \dots - x_d^2}$

$$\int d^d x\, e^{-x^2} = \prod_{i=1}^d \int dx_i\, e^{-x_i^2} = \pi^{d/2}$$

On the other hand

$$\pi^{d/2} = S_d \int_0^\infty dx\, x^{d-1} e^{-x^2} = [x = \sqrt{y}] =$$

$$= \frac{S_d}{2} \int_0^\infty \frac{dy}{y^{1/2}}\, y^{\frac{d-1}{2}} e^{-y} =$$

$$= \frac{S_d}{2} \int_0^\infty dy\, y^{d/2-1} e^{-y} = \frac{S_d}{2} \Gamma\left(\frac{d}{2}\right)$$

$$S_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \Leftarrow$$

$$\langle (\delta\varphi)^2 \rangle = \sum_{q_0/2<|q|<q_0} \frac{1}{cq^2} = \underbrace{\frac{2\pi^{d/2}}{(2\pi)^d\, \Gamma\left(\frac{d}{2}\right)}}_{K_d} \int_{q_0/2}^{q_0} \frac{dq\, q^{d-1}}{cq^2} = \frac{K_d}{c} \int_{q_0/2}^{q_0} q^{d-3}\, dq =$$

$$K_d = \frac{1}{2^{d-1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)}$$

$$= \frac{K_d}{c(d-2)} \left[q_0^{\,d-2} - \left(\frac{q_0}{2}\right)^{d-2}\right] = \frac{K_d}{c(d-2)}\, q_0^{\,d-2} \left[1 - 2^{2-d}\right]$$

$$\boxed{\langle (\delta\varphi)^2 \rangle = n_c \left(1 - \ell^{2-d}\right)} \qquad n_c = \frac{K_d\, q_0^{d-2}}{c(d-2)}$$

$$\langle \mathcal{H}_1 \rangle = u \int d^dx \, \langle \varphi^4 \rangle = u \int d^dx \, \langle (\varphi' + \delta\varphi)^4 \rangle =$$

$$= u \int d^dx \left(\varphi'^4 + 6\varphi'^2 \langle \delta\varphi \rangle^2 + \langle \delta\varphi \rangle^4 \right) = \left\{ \begin{matrix} \langle \delta\varphi \rangle = 0 \\ \langle \delta\varphi^3 \rangle = 0 \end{matrix} \right\}$$

$$= u \int d^dx \, \varphi'^4 + 6u \int d^dx \, \varphi'^2 \langle \delta\varphi \rangle^2 + 3u \int d^dx \, \langle (\delta\varphi)^2 \rangle^2$$

$$\langle \mathcal{H}_0 \rangle = \frac{1}{2} c \int d^dx \, (\nabla\varphi)^2 + \frac{1}{2} r \int d^dx \, \varphi^2 = \frac{1}{2} c \int d^dx \, (\nabla\varphi')^2 +$$

$$+ \frac{1}{2} r \int d^dx \, \varphi'^2 + \frac{1}{2} \sum_{q_0/\ell < q < q_0} (cq^2 + r) \langle (\delta\varphi_q)^2 \rangle$$

$$\mathcal{H}_0 : \frac{1}{2} r \varphi'^2 \qquad \mathcal{H}_0 + \langle \mathcal{H}_1 \rangle : \frac{1}{2} r \varphi'^2 + 6u \varphi'^2 \langle (\delta\varphi)^2 \rangle$$

$$r \to r + 12u \langle (\delta\varphi)^2 \rangle = r + 12 u n_c \left(1 - \ell^{2-d}\right)$$

__No__ new terms of $\underline{(\nabla\varphi)^2}$ in the <u>first order</u> in u.

rescaling we do in the way $\int d^dx \, c(\nabla\varphi)^2$ is invariant

$$\vec{x}' = \vec{x}/\ell \qquad \varphi(\vec{x}) \to \varphi'(\vec{x}')\, \ell^{1-d/2}$$

$$\int d^dx \, c(\nabla\varphi)^2 \longrightarrow \int d^dx' \, c(\nabla'\varphi')^2$$

In the first order in u: $\left\{ \begin{aligned} r' &= \ell^2 \left[r + 12 u n_c \left(1 - \ell^{2-d}\right) \right] \\ u' &= \ell^{4-d} u \end{aligned} \right.$

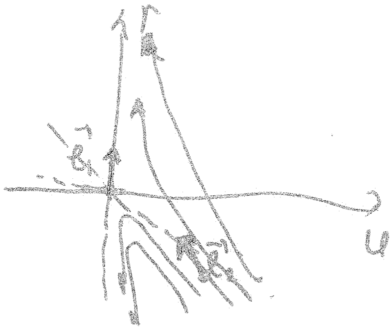
Immediately: $\eta = 0$ in the first order in u.

$$\begin{pmatrix} r' \\ u' \end{pmatrix} = \hat{R} \begin{pmatrix} r \\ u \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} \lambda^2 & B(\lambda^2 - \lambda^{4-d}) \\ 0 & \lambda^{4-d} \end{pmatrix} \qquad B = 12 n_c$$

$$y_1 = 2 \qquad y_2 = 4-d \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{e}_2 = \begin{pmatrix} -B \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+B^2}}$$

$d > 2 \quad y_1 > y_2 \qquad \vec{e}_1 \cdot \vec{e}_2 \neq 0$

(I) $d > 4 \Rightarrow y_2 < 0,\ y_1 > 0$

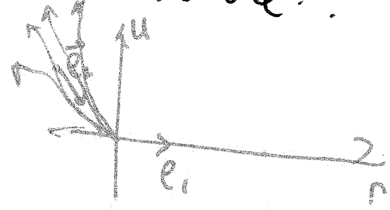


Gaussian fixed point $r=0, u=0, c = \text{const}$

$\lambda > 1$, λ is real

The trajectories ~~will~~ be attracted by y-axis

(II) $d < 4$? Gaussian fixed point is unstable!!!



Requirment: $y_1 > 0 \Rightarrow \nu = \frac{1}{y_1}$

Stable fixed point

$\mathrm{Re}\, y_2 < 0 \quad \mathrm{Re}\, y_3 < 0 \Rightarrow$ trajectory will go to r-axis

Then our system is reproduced in any step of renormalization

Gaussian point is stable at $d > 4$. It is unstable $d < 4$.

GFP: $\eta = 0 \quad \nu = 1/2$ — No additional $(\nabla\varphi)^2$ $\quad y_1 = 2 \quad y_t < 0$

The prediction of scaling — Ornstein-Zernike theory $\nu = \frac{1}{y_1}$

$\alpha = 2 - \nu d = 2 - d/2$ $\qquad \mathcal{H} = \frac{1}{2}\int d^d x\, (\nabla\varphi)^2 + r\varphi^2 - \int d^d x\, h\varphi$

? $\beta = \frac{\nu}{2}(d-2+\eta) = \frac{d-2}{4}$ $\qquad \alpha = 2 - \frac{d}{2}$

$\gamma = \nu(2-\eta) = 1 - \frac{\eta}{2}$ $\qquad \beta = \frac{1}{2}$ (Landau theory)

? $\delta = \frac{d+2-\eta}{d-2+\eta} = \frac{d+2}{d-2}$ $\qquad \gamma = 1$, $\quad \delta = 3$ (Landau theory)

~~On~~ At $d>4$ scaling hypothesis is wrong!!!

Only at $d=4$ $\Leftrightarrow$ coincide.

You should use RG to find a form of Hamiltonian.

Scaling hypothesis ~~w~~ works at $d<4$.

Relevant and irrelevant variables:

$$\mathcal{H} = \frac{1}{2}\int d^d x \, c(\nabla\varphi)^2 + r\varphi^2 \cancel{+ u\varphi^4(x)} + \cancel{v\int d^d x\, \varphi^6(x)} + \cancel{v'\int d^d x (\nabla\varphi)^2\varphi^2} \dots$$

RG gives an answer which term are important!

$$\vec{x} \to \vec{x}\,\lambda \qquad \varphi \to \lambda^a \varphi$$

$$r \to \lambda^{d+2a} r \qquad c \to \lambda^{d-2+2a} c \qquad u \to \lambda^{d+4a} u \qquad v \to \lambda^{d+6a} v$$

$c \to c$

$d-2+2a=0$

$a = -\frac{d-2}{2}$ | $\eta = 0$ at crit point

$$r \to \lambda^2 r \qquad u \to \lambda^{-d+4} u \qquad v \to \lambda^{6-2d} v$$

$d>4$ $\quad u \to 0$ $\quad \nu = \frac{1}{2}$

$d>3$ $\quad v \to 0$

The effective hamiltonian to describe behavior at crit. point

$d>4$ $$\mathcal{H} = \frac{1}{2}\int d^d x \, c(\nabla\varphi)^2 + \frac{1}{2}\int d^d x \, r\varphi^2$$